A Dobrushin–Major–Taqqu Theorem for Hilbert Space-Valued Random Variables

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November 11, 2026

Abstract

Let $\{X_t : t \in \mathbb{Z}\}$ be a stationary Gaussian process taking values in a separable Hilbert space $\mathbb{H}_1$, with long-range dependence defined in the spectral domain through a memory operator $A : \mathbb{H}_1 \to \mathbb{H}_1$. For square integrable, nonlinear operators $G : \mathbb{H}_1 \to \mathbb{H}_2$, taking values in another separable Hilbert space $\mathbb{H}_2$, we establish a non-central limit theorem for the centered partial sums process of $\{G[X_n]\}_n$, whenever $G[X_n]$ inherits the long-range dependence behavior from its Gaussian input process. We identify an operator normalization that accounts for both the underlying memory structure, characterized by the memory operator A, and the nonlinear transformation G. The limiting Hilbert space valued process is, in general, non Gaussian and can be expressed as multiple Wiener integrals involving only finitely many Wiener chaos orders along fixed directions in $\mathbb{H}_2$. The proof introduces a Hilbert-space valued generalization of the reduction principle, identifying the dominant Hermite components and showing that the remainder is negligible under our operator normalization. We employ our results to prove non-central limit theorems for nonlinear transformations of functional FARIMA processes, and sojourn functionals of temporally dependent spherical Gaussian fields.

1 Introduction

Let $\{X_t : t \in \mathbb{Z}\}$ be a centered stationary Gaussian sequence with values in a separable Hilbert space $\mathbb{H}_1$ and $G : \mathbb{H}_1 \to \mathbb{H}_2$ a measurable map, with $\mathbb{H}_2$ another separable Hilbert space. Define the partial-sum process

$$S_n(s) = T_n \sum_{t=1}^{\lfloor ns \rfloor} \left(G[X_t] - \mathrm{E}\, G[X_t] \right), \qquad s \in [0,1], \tag{1.1}$$

where $\{T_n\}_n$ is a suitable sequence of scaling operators. This work aims to understand the weak limit of $\{S_n\}_n$ in (1.1) when the process $\{G[X_n]\}_n$ exhibits long-range dependence.

The complementary regime, in which $\{G[X_n]\}_n$ exhibits short-range dependence, is customarily referred to as the Breuer–Major setting, in recognition of the seminal work [7] in the case $\mathbb{H}_1 = \mathbb{H}_2 = \mathbb{R}$; see also [19]. In this case, the process (1.1) falls within the Gaussian universality class, with the usual scaling $T_n = n^{-1/2} I_{\mathbb{H}_2}$. More recently, a line of work based on Malliavin calculus and Stein's method has provided new proofs and extensions of such central limit theorems, e.g., [9, 38, 39]. Using the Malliavin–Stein framework in infinite-dimensional spaces [15], two of the present authors recently established the first extension of the Breuer–Major theorem to the fully Hilbert-space setting considered here [17].

In the long-range dependent regime, the limit of (1.1) is generally non-Gaussian and such weak convergence results are referred to as non-central limit theorems. In the scalar case $\mathbb{H}_1 =$

$\mathbb{H}_2 = \mathbb{R}$, the corresponding non-central limit theorems are due to Dobrushin and Major [13] and Taqqu [45, 46], whose work motivates the title of the present article. In this case, the scaling T_n depends, in general, on the dependence properties of $\{G(X_k)\}$, and the partial sums $\{S_n\}$ fall under the universality classes of the so-called Hermite processes. The cases $\mathbb{H}_1 = \mathbb{R}^d, \mathbb{H}_2 = \mathbb{R}$ and $\mathbb{H}_1 = \mathbb{R}, \mathbb{H}_2 = \mathbb{R}^d$ were later studied in [1, 44] and [2], respectively. This topic continues to receive attention; see, for instance, [32] for the case $\mathbb{H}_1 = \mathbb{R}^d, \mathbb{H}_2 = \mathbb{R}$, with a latent Gaussian random field $\{X_k\}_{k\in\mathbb{Z}^r}$ indexed by the lattice $\mathbb{Z}^r$. Finally, Theorem 2.7 of [4] studies the case $\mathbb{H}_1 = \mathbb{R}, \mathbb{H}_2 = L^2(\mathbb{R}, w(t)dt)$, where w is a suitable weight function, and T_n is a scalar multiple of the identity times a slowly varying function, with the scalar normalization determined by the leading term. Their result can be recovered as a special case of ours, modulo the presence of the slowly varying function, giving rise to a substantially simpler limiting process.

We establish a non-central limit theorem allowing both the input and output spaces to be possibly infinite-dimensional, general Hilbert spaces. Under spectral assumptions on $\{X_k\}$ (cf. Assumption 3.1) and mild assumptions on the map G (cf. Assumptions 3.2), we determine the limit of (1.1), with $\{T_n\}_n$ a suitable sequence of scaling operators. In this setting, different projections in $\mathbb{H}_2$ may require different normalizations, and different Wiener chaos orders may contribute to the limit, even within a single direction. We identify the operator normalization $T_n = n^{-D}$ through the effective memory exponents of the active Hermite terms and represent the limiting process by a series of Hilbert-valued multiple Wiener integrals.

Our motivation for this problem stems from the asymptotic analysis of sojourn measures of Gaussian random fields exhibiting long-range dependence. To be more precise, consider a Gaussian random field $Z = \{Z(x,t) : x \in \mathcal{D},\ t \in \mathcal{T}\}$, where $\mathcal{D}$ is a spatial domain such as $[0,T]^d$ or the sphere $\mathbb{S}^2$ [35], $\mathcal{T}$ is a time index set such as $\mathbb{R}$ or $\mathbb{Z}$, and $\{Z(\cdot,t)\}_{t\in\mathcal{T}}$ is stationary and exhibits long-range dependence. Viewing $Z(\cdot,t)$ as an $L^2(\mathcal{D})$-valued random element, set $\mathbb{H}_1 = L^2(\mathcal{D})$ and $\mathbb{H}_2 = \mathbb{R}$. For a measurable domain $D \subseteq \mathcal{D}$ of finite measure, the centered sojourn functional above u is, for $k \in \mathcal{T}$,

$$\mathcal{S}_D^k(u) = G_u[Z(\cdot,k)] - \mathrm{E}\, G_u[Z(\cdot,0)], \qquad G_u[f] \ = \int_D \mathbb{1}_{\{f(x)\geq u\}}\, dx = \left|\{x \in D : f(x) \geq u\}\right|.$$

where $|\cdot|$ denotes Lebesgue or surface measure, as appropriate. The associated asymptotic problem concerns the normalization and limiting distribution of centered sojourn measures as the observation domain expands. For a time-indexed spatial field, this also leads to the study of temporal sums of $G_u[Z(\cdot,t)] - \mathrm{E}\, G_u[Z(\cdot,0)]$.

These questions have been investigated under both short- and long-range dependence; see, for example, [8, 22, 25, 26, 33, 35]. A central tool in this literature is the reduction principle, which identifies the chaos components governing the asymptotic fluctuations through Hermite or Laguerre expansions. Classical results [13, 29, 30, 45, 46] establish this approach for long-range dependent Gaussian and chi-squared subordinated fields. In the spatiotemporal setting, Leonenko and Ruiz–Medina [27, 28] apply reduction principles to obtain central limit theorems under increasing-domain asymptotics in time and/or space. Marinucci, Rossi and Vidotto [34] derive central and non-central limit theorems for sojourn measures of stationary and spatially isotropic Gaussian fields on the sphere under increasing-time asymptotics, showing how temporal dependence and the excursion threshold determine the normalization and limiting law. Their spherical-harmonic formulation makes the role of the angular spectrum explicit. Extensions to chi-squared subordinated fields, using tensor products of Hermite polynomial bases, are developed in [10]. See Section ref 7.2 for the application of our result to such a setting.

Although we are the first to treat the partial sums of subordinated Gaussian sequences (1.1) at this level of generality, several works establish limit theorems for stationary Hilbert-valued sequences. The theory is more developed for linear series [6, 36, 37], and a few works also study limit theorems for long-range dependent linear series in Hilbert spaces [11, 14, 16]. In particular, the last of these is the first work we are aware of to construct a non-Gaussian Hermite process taking values in an infinite-dimensional Hilbert space.

Our main result, Theorem 3.1, establishes a non-central limit theorem under spectral assumptions, with the anticipated operator normalization n^{-D} on $\mathbb{H}_2$. A key assumption is that a single orthonormal basis of $\mathbb{H}_1$, denoted by $\{\phi_i\}_{i\in\mathbb{N}}$, simultaneously diagonalizes all covariance and spectral density operators. Moreover, we assume a semiparametric characterization of the spectral densities in this basis, such that each coordinate exhibits long-range dependence, with uniform control of the associated memory parameters and slowly varying functions; see Assumption 3.1. Our assumptions on the functional time series $\{X_k\}_k$ are satisfied by Gaussian functional FARIMA models, including models with a common memory parameter [31] and models with coordinate-dependent memory parameters [18, 43], under suitable summability and uniform spectral bounds. The simultaneous diagonalization condition also holds for fields in statistical physics whose spatial domain is a connected and compact two-point homogeneous space, provided that the covariance and spectral density operators are invariant under the isometry group of the manifold; see, for example, [40].

The proof relies on a Wiener–Itô chaos decomposition for Hilbert-valued random variables (see, for instance, Sections 2 and 4 of [17]) and is split into two parts. First, we prove a reduction principle (cf. Proposition 5.2). In each basis direction v_i of $\mathbb{H}_2$, the fluctuations are governed by the leading terms, which share an effective memory exponent determined by their Hermite multi-indices and the memory parameters of the underlying Gaussian sequence. For the proof of the reduction principle, we use finite Hermite projections and uniform bounds on the projection remainders. Second, we prove that the kernels of the leading terms converge to the corresponding kernels of the limiting process (4.9) (cf. Proposition 5.1). Each individual term in this representation belongs to a fixed Wiener chaos. We establish the kernel convergence for each fixed Hermite multi-index and extended to the full expansion by dominated convergence. Moreover, the limiting process need not belong to a single Wiener chaos, since different chaos orders may survive within a single direction of $\mathbb{H}_2$, as well as across different directions.

The paper is organized as follows. Section 2 introduces the spectral setting and some preliminaries for Hilbert-valued random variables. In Section 3, we state and discuss our assumptions and present the main operator-normalized theorem. Section 4 constructs the limiting process appearing in our theorem. Section 5 contains the kernel convergence result and the reduction principle, together with their proofs and the proof of the main theorem. Finally, Section 7 gives two applications of our result.

2 Preliminaries

In this section we present some preliminary facts about random variables taking values in Hilbert spaces, as well as their Hermite and Wiener chaos expansions.

2.1 Random variables on Hilbert spaces

Let $\mathbb{H}$ be a real separable Hilbert space with the inner product $\langle\cdot,\cdot\rangle_{\mathbb{H}}$, and denote by $\widetilde{\mathbb{H}} = \mathbb{H}+i\mathbb{H}$, its complex version, whose elements are functions of the form $\psi = \psi_1 + i\psi_2$ for $\psi_i \in \mathbb{H}$, $i = 1, 2$, equipped with the inner product given by

$$\left\langle \psi_1 + i\psi_2, \widetilde{\psi}_1 + i\widetilde{\psi}_2 \right\rangle_{\widetilde{\mathbb{H}}} = \left\langle \psi_1, \widetilde{\psi}_1 \right\rangle_{\mathbb{H}} + \left\langle \psi_2, \widetilde{\psi}_2 \right\rangle_{\mathbb{H}} + i\left(\left\langle \psi_2, \widetilde{\psi}_1 \right\rangle_{\mathbb{H}} - \left\langle \psi_1, \widetilde{\psi}_2 \right\rangle_{\mathbb{H}} \right).$$

We denote by $(\Omega, \mathcal{F}, P)$ the underlying probability space, and by $L^2(\Omega, \mathcal{F}, P : \mathbb{H})$ the space of square integrable random variables taking values in $\mathbb{H}$, where $\mathbb{H}$ may denote a real or complex Hilbert space. For compact operators on $\mathbb{H}$ (where again $\mathbb{H}$ may be real or complexified), fix a basis $\{\phi_i\}_i$ of $\mathbb{H}$; let $\mathcal{S}_1(\mathbb{H})$, $\mathcal{S}_2(\mathbb{H})$, and $\mathcal{L}(\widetilde{\mathbb{H}})$ denote the trace-class, Hilbert–Schmidt, and operator norm classes, respectively. For an operator $\mathcal{A}$ with adjoint $\mathcal{A}^\star$ the corresponding norms

are given by

$$\|\mathcal{A}\|_{\mathcal{S}_1(\mathbb{H})} = \sum_{k\geq 1} \left\langle [\mathcal{A}^\star \mathcal{A}]^{1/2}(\psi_k), \psi_k \right\rangle_{\mathbb{H}},$$

$$\|\mathcal{A}\|_{\mathcal{S}_2(\mathbb{H})} = \sqrt{\|\mathcal{A}^\star \mathcal{A}\|_{\mathcal{S}_1(\mathbb{H})}}, \qquad \|\mathcal{A}\|_{\mathcal{L}(\mathbb{H})} = \sup_{\psi\in\mathbb{H};\ \|\psi\|_{\mathbb{H}}=1} \|\mathcal{A}(\psi)\|_{\mathbb{H}},$$

Recall that $\{X_k : k \in \mathbb{Z}\}$ denotes a centered, stationary, Gaussian, and $\mathbb{H}_1$-valued random variables, with covariance operator Q, and denote the law of X_0 by γ_Q. Its autocovariance operators are

$$\Gamma(v) = \mathrm{E}[X_v \otimes X_0], \qquad \Gamma(v)h = \mathrm{E}\left[\langle X_0, h\rangle_{\mathbb{H}_1} X_v\right], \qquad v \in \mathbb{Z},$$

such thay $Q = \Gamma(0)$. From the spectral theorem, there is a basis $\{\phi_r\}_r$ of $\mathbb{H}_1$ such that $Q\phi_r = \lambda_r\phi_r$ for all $r \geq 1$, where $\sum_{r\geq 1}\lambda_r = \mathrm{Tr}(Q)$. Without loss of generality, we can assume that Q is nondegenerate (meaning $\langle Qh, h\rangle_{\mathbb{H}_1} > 0$ for all $h \neq 0$) by excluding the coordinates ϕ_r with $\lambda_r = 0$ from $\mathbb{H}_1$.

Following [41] and [43], we define by $\mathscr{F}_\omega$ the spectral density of the Hilbert-space valued process $\{X_k\}_k$. Assume that $\mathscr{F}_\omega$ is a strongly measurable positive self-adjoint trace-class operator for almost every $\omega \in [-\pi, \pi]$. Under Assumption 3.1 below, the autocovariance operators are given by

$$\Gamma(t) = \int_{-\pi}^{\pi} e^{it\omega}\mathscr{F}_\omega \, d\omega, \qquad t \in \mathbb{Z},$$

where these integrals are understood in the Bochner sense and imply that $\Gamma(0) \in \mathcal{S}_1(\mathbb{H}_1)$, $\mathrm{E}\,\|X_0\|^2_{\mathbb{H}_1} < \infty$, and that

$$\|\Gamma(t)\|_{\mathcal{S}_1(\mathbb{H}_1)} \leq \int_{-\pi}^{\pi} \mathrm{Tr}(\mathscr{F}_\omega)\ d\omega.$$

Moreover, the Banach-valued Riemann–Lebesgue lemma gives $\|\Gamma(t)\|_{\mathcal{S}_1(\mathbb{H}_1)} \to 0$ as $|t| \to \infty$. Under Assumption 3.1, there is a common basis $\{\phi_k\}$ diagonalizing the spectral operator family $\{\mathscr{F}_\omega\}_{\omega\in[-\pi,\pi]}$ for almost all $\omega \in [-\pi, \pi]$, admitting the family of eigenvalues $\{f_k(\omega)\}_{k\in\mathbb{N},\omega\in[-\pi,\pi]}$. As a result of Assumptions 3.1(i)–(ii), the same basis diagonalizes $\{\Gamma(t)\}_{t\in\mathbb{Z}}$, the eigenvalues $\{\lambda_r\}_r$ of Q are related to $\{f_k(\omega)\}_{k\in\mathbb{N},\omega\in[-\pi,\pi]}$ by $\lambda_r = \int_{-\pi}^{\pi} f_r(\omega)\ d\omega$, and we obtain

$$\sum_{k\geq 1}\int_{-\pi}^{\pi} f_k(\omega)\ d\omega < \infty.$$

Finally, we define the normalized spectral eigenvalues by

$$\widetilde{f}_r(\omega) = \frac{f_r(\omega)}{\lambda_r}, \quad \omega \in [-\pi, \pi], \qquad \text{so that} \int_{-\pi}^{\pi} \widetilde{f}_r(\omega)\ d\omega = 1,$$

2.2 The infinite-dimensional Hermite expansion

In this section, we collect some facts about the white noise and Hermite expansions for random variables taking values in $\mathbb{H}_1$. We mostly use the exposition of [17].

Define the white-noise mapping by, for $x, h \in \mathbb{H}_1$,

$$W_h(x) = \sum_{r\geq 1} \lambda_r^{-1/2}\langle x, \phi_r\rangle_{\mathbb{H}_1}\langle h, \phi_r\rangle_{\mathbb{H}_1}, \qquad \text{implying } W_{\phi_r}(X_k) = \frac{\langle X_k, \phi_r\rangle_{\mathbb{H}_1}}{\sqrt{\lambda_r}}. \tag{2.1}$$

where the series converges in $L^2(\mathbb{H}_1, \gamma_Q)$.

Let $G : \mathbb{H}_1 \to \mathbb{H}_2$ be centered and in $L^2(\mathbb{H}_1, \gamma_Q : \mathbb{H}_2)$, i.e., $\mathrm{E}\,\|G[X_0]\|^2_{\mathbb{H}_2} < \infty$. Denote with $\{v_i : i \geq 1\}$ an orthonormal basis of $\mathbb{H}_2$.

The probabilists' Hermite polynomials are defined by $H_0(z) = 1$ and, for $p \geq 1$, $H_p(z) = (-1)^p e^{z^2/2} \frac{d^p}{dz^p} e^{-z^2/2}$. For standard jointly Gaussian U, V, $\mathrm{E}[H_p(U)H_m(V)] = p!(\mathrm{E}[UV])^p \delta_{pm}$. Set

$$\mathcal{L} \doteq \Big\{\ell = (\ell_m)_{m\geq 1} \in \mathbb{N}_0^\infty : |\ell| \doteq \sum_{m\geq 1} \ell_m < \infty\Big\}, \quad \mathcal{L}_p \doteq \{\ell \in \mathcal{L} : |\ell| = p\}, \quad \ell! \doteq \prod_{m\geq 1} \ell_m!, \tag{2.2}$$
$$M_0 \doteq 0, \qquad M_\ell \doteq \max\{m : \ell_m \geq 1\} \quad (\ell \neq 0).$$

The generalized Hermite polynomials and coefficients are

$$H_\ell[x] \doteq \prod_{m=1}^{M_\ell} H_{\ell_m}(W_{\phi_m}(x)), \qquad \ell \in \mathcal{L},$$

with the empty product equal to one. Then, from Lemma 2.1 in [17], we have the Hermite decomposition

$$G[x] = \sum_{i\geq 1} \sum_{\ell\in\mathcal{L}} c_{i,\ell} H_\ell[x] \otimes v_i, \quad \text{where} \quad c_{i,\ell} \doteq \frac{1}{\ell!} \langle G, H_\ell \otimes v_i\rangle_{L^2(\mathbb{H}_1,\gamma_Q)\otimes\mathbb{H}_2} \tag{2.3}$$

where we have used the identification $L^2(\mathbb{H}_1, \gamma_Q : \mathbb{H}_2) \cong L^2(\mathbb{H}_1, \gamma_Q)\otimes\mathbb{H}_2$. The notation $H_\ell[x]\otimes v_i$ is identified with usual multiplication $H_\ell[x]v_i$.

Lemma 2.1. *The family* $\{(\ell!)^{-1/2} H_\ell \otimes v_i : \ell \in \mathcal{L},\ i \geq 1\}$ *is an orthonormal basis of* $L^2(\mathbb{H}_1, \gamma_Q : \mathbb{H}_2)$. *In particular,* (2.3) *converges in that Hilbert space and* $\mathrm{E}\,\|G[X_0]\|^2_{\mathbb{H}_2} = \sum_{i\geq 1}\sum_{\ell\in\mathcal{L}} \ell! |c_{i,\ell}|^2$.

For nonzero centered G, its Hermite rank and chaotic expansion are defined by

$$q = \mathrm{Rank}(G) \doteq \min\{p \geq 1 : c_{i,\ell} \neq 0 \text{ for some } i \geq 1,\ \ell \in \mathcal{L}_p\},$$

$$G = \sum_{p=q}^{\infty} G_p, \qquad G_p[x] \doteq \sum_{i\geq 1} \sum_{\ell\in\mathcal{L}_p} c_{i,\ell} H_\ell[x] \otimes v_i,\ p \geq q, \tag{2.4}$$

and taking $G_0 = \cdots = G_{q-1} = 0$, where G_p denotes the chaos projection. Moreover, by the orthogonality of the Hermite polynomials we get the identities

$$\mathrm{E}\,\|G[X_0]\|^2_{\mathbb{H}_2} = \sum_{p=q}^{\infty}\sum_{i=1}^{\infty}\sum_{\boldsymbol{\ell}\in\mathcal{L}_p} \boldsymbol{\ell}!\,|c_{i,\boldsymbol{\ell}}|^2 < \infty, \quad \mathrm{E}\,\|G_q[X_0]\|^2_{\mathbb{H}_2} = \sum_{i=1}^{\infty}\sum_{\boldsymbol{\ell}\in\mathcal{L}_q} \boldsymbol{\ell}!\,|c_{i,\boldsymbol{\ell}}|^2 < \infty.$$

2.3 Wiener chaos expansions on Hilbert spaces

Let $\widehat{W}$ be a Hermitian Gaussian spectral measure on $\mathbb{R}\times\mathbb{N}$, with control measure $dx\otimes\#$, so that its coordinate measures $\widehat{W}_j$ are independent and $\widehat{W}_j(-A) = \overline{\widehat{W}_j(A)}$. Let $\mathfrak{H}$ be the real Hilbert space of complex-valued $L^2(dx\otimes\#)$ functions satisfying $f(-x,j) = \overline{f(x,j)}$, with the real L^2 inner product.

For $p \geq 1$, let I_p denote the multiple Wiener–Itô integral of order p with respect to $\widehat{W}$. Equip the symmetric tensor powers with the norm inherited from the full tensor power, keeping the factorial explicitly in the isometry. For $f \in \mathfrak{H}^{\odot p}$ and $g \in \mathfrak{H}^{\odot q}$, we then have

$$\mathrm{E}[I_p(f)I_q(g)] = \begin{cases} p!\langle f, g\rangle_{\mathfrak{H}^{\otimes p}}, & p = q, \\ 0, & p \neq q. \end{cases} \tag{2.5}$$

We denote its Hilbert-valued counterpart by $\mathcal{I}_p = I_p \otimes \mathrm{Id}_{\mathbb{H}_2}$.

Since the standardized coordinate process $\{W_{\phi_j}(X_t)\}_{t\in\mathbb{Z}}$ has spectral density $\widetilde{f}_j$, we may work with a joint spectral realization such that

$$W_{\phi_j}(X_t) = \int_{-\pi}^{\pi} e^{it\omega}\widetilde{f}_j(\omega)^{1/2}\widehat{W}_j(d\omega) = I_1(h_{j,t}), \quad h_{j,t}(\omega,t) \doteq e^{it\omega}\widetilde{f}_j(\omega)^{1/2}\mathbb{1}_{\{[-\pi,\pi]\}}(\omega)\mathbb{1}_{\{r=j\}}. \tag{2.6}$$

Let Sym_p denote the standard symmetrization operator from $\mathfrak{H}^{\otimes p}$ onto $\mathfrak{H}^{\odot p}$; see [39, Appendix B.3]. For $\boldsymbol{\ell} \in \mathcal{L}_p$, let $k_1 \le \cdots \le k_p$ list the coordinate indices, with each j appearing exactly ℓ_j times. For a square-integrable kernel f satisfying $f(-x) = \overline{f(x)}$, define

$$f_{\boldsymbol{\ell}}((x_1,j_1),\dots,(x_p,j_p)) = f(x_1,\dots,x_p)\prod_{a=1}^{p}\mathbb{1}_{\{j_a=k_a\}}, \quad J_{\boldsymbol{\ell}}f = \mathrm{Sym}_p f_{\boldsymbol{\ell}}. \tag{2.7}$$

If f is invariant under permutations of positions carrying the same label, then

$$p!\|J_{\boldsymbol{\ell}}f\|^2_{\mathfrak{H}^{\otimes p}} = \boldsymbol{\ell}!\|f\|^2_{L^2(\mathbb{R}^p)}. \tag{2.8}$$

From (2.6), the labelled kernels satisfy $\|h_{j,t}\|_{\mathfrak{H}} = 1$ and $\langle h_{j,t}, h_{k,t}\rangle_{\mathfrak{H}} = 0$ for $j \neq k$. Thus $H_{\ell_j}(W_{\phi_j}(X_t)) = I_{\ell_j}(h_{j,t}^{\otimes \ell_j})$, and all cross-coordinate contractions vanish in the product formula. Expanding the resulting tensor product and applying $J_{\boldsymbol{\ell}}$ therefore yields

$$H_{\boldsymbol{\ell}}[X_t] = I_p\left(J_{\boldsymbol{\ell}}\left[e^{it\sum_{a=1}^{p}\omega_a}\prod_{a=1}^{p}\left(\widetilde{f}_{k_a}(\omega_a)^{1/2}\mathbb{1}_{\{|\omega_a|\le\pi\}}\right)\right]\right), \tag{2.9}$$

where the spectral densities are extended by zero outside $[-\pi,\pi]$.

3 Assumptions and Main results

In this section, we present our assumptions, main results, and a discussion on the assumptions.

3.1 Assumptions

We start with the central assumption of our work. We call an input coordinate j *active* if it appears in a nonzero term of the Hermite expansion of G, i.e., there is a a multi-index ℓ and an i such that $\ell_j > 0$ and $c_{i,\ell} \neq 0$.

We now state the spectral assumptions on the underlying process $\{X_k\}_k$.

Assumption 3.1. Assume that the following hold for the spectral density operators $\{\mathscr{F}_\omega\}_\omega$:

(i) For almost every $\omega \in [-\pi,\pi]$, $\mathscr{F}_\omega \in \mathcal{S}_1(\widetilde{\mathbb{H}}_1)$ and $\int_{-\pi}^{\pi}\mathrm{Tr}(\mathscr{F}_\omega)\,d\omega < \infty$.

(ii) There is an orthonormal basis $\{\phi_k : k \ge 1\}$ of $\mathbb{H}_1$ such that, for all $k \ge 1$,

$$\mathscr{F}_\omega\phi_k = f_k(\omega)\phi_k, \qquad f_k(\omega) \ge 0, \quad f_k(-\omega) = f_k(\omega), \tag{3.1}$$

for almost every $\omega \in [-\pi,\pi]$.

(iii) For every $k \in \mathbb{N}$, there exist constants $0 < \alpha_k \le \alpha < 1$, $\kappa_k > 0$, and $M \ge 0$, and a measurable function $L_k : (0,\pi] \to [0,\infty)$, such that

$$f_k(\omega) = \lambda_k\kappa_k|\omega|^{-\alpha_k}L_k(|\omega|), \qquad \text{for a.e. } 0 < |\omega| \le \pi, \tag{3.2}$$

where $\lim_{r\downarrow 0} L_k(r) = 1$ and $\sup_k L_k(r) \le M$ a.e. $0 < r \le |\pi|$.

(iv) Assume that the standardized spectral scales are uniformly bounded:

$$\kappa_* \doteq \sup\{\kappa_k : k \text{ is active}\} < \infty.$$

We state now an important consequence of Assumption 3.1. Recall the memory parameters $\{\alpha_j\}_j$ from Assumption 3.1(iii). For $\boldsymbol{\ell} \in \mathcal{L}_p$, define

$$\beta(\boldsymbol{\ell}) = \sum_j \ell_j(1-\alpha_j), \qquad H(\boldsymbol{\ell}) = 1 - \tfrac{1}{2}\beta(\boldsymbol{\ell}). \tag{3.3}$$

The quantity $\beta(\boldsymbol{\ell})$ combines the coordinate memory parameters with their Hermite multiplicities, and we call $H(\boldsymbol{\ell})$ the *effective exponent* of the corresponding Hermite term. The effective exponent corresponds to the scaling of our sums. We call the index $H(\ell)$ *noncentral* when $H(\ell) > 1/2$ or, equivalently, $\beta(\ell) < 1$. Denoting $\varepsilon = 1 - \sup_j \alpha_j > 0$, this implies that $\beta(\boldsymbol{\ell}) \geq \varepsilon \sum_j \ell_j = p\varepsilon$, hence $p < \varepsilon^{-1}$. In other words, for each fixed ε, we can only obtain Hermite polynomials of order up to $p < \varepsilon^{-1}$.

Define the dominating index and sets respectively by

$$\begin{gathered} d_i = \sup\{H(\boldsymbol{\ell}) : c_{i,\boldsymbol{\ell}} \neq 0,\ H(\boldsymbol{\ell}) > 1/2\}, \\ \mathcal{D}_i = \{\boldsymbol{\ell} \in \mathcal{L} : c_{i,\boldsymbol{\ell}} \neq 0,\ H(\boldsymbol{\ell}) = d_i\}, \quad \mathcal{N}_i = \{\boldsymbol{\ell} \in \mathcal{L} : c_{i,\boldsymbol{\ell}} \neq 0\} \setminus \mathcal{D}_i, \end{gathered} \tag{3.4}$$

giving rise to the dominating part of the Hermite expansion

$$G_{\mathcal{D}}[X_t] = \sum_{i=1}^{\infty} \sum_{\boldsymbol{\ell} \in \mathcal{D}_i} c_{i,\boldsymbol{\ell}} H_{\boldsymbol{\ell}}[X_t] v_i. \tag{3.5}$$

Assumption 3.2. Fix a measurable, centered map $G : \mathbb{H}_1 \to \mathbb{H}_2$ with coefficients as in (2.3) and assume the following:

(i) For every nonzero output coordinate, assume that its set of active noncentral indices is nonempty and that the supremum defining d_i in (3.4) is attained. For an identically zero output coordinate, set $d_i = 3/4$ and $\mathcal{D}_i = \varnothing$.

(ii) The following is satisfied

$$\sum_{i=1}^{\infty} \frac{\mathrm{E}\,|\langle G[X_0], v_i\rangle_{\mathbb{H}_2}|^2}{2d_i - 1} < \infty.$$

An important corollary of our main setting is the one where there is a common active component. In this simplified setting, Assumption 3.2 will be replaced by the following, simplified hypothesis.

Assumption 3.3. Suppose the following three assumptions hold:

(i) The map $G \in L^2(\mathbb{H}_1, \gamma_Q : \mathbb{H}_2)$ has hermite rank q.

(ii) There is $\alpha \in (0,1)$ such that $c_{i,\boldsymbol{\ell}} \neq 0$ for $\boldsymbol{\ell} \in \mathcal{L}_q$ implies $\alpha_j = \alpha$ for all $j \in \operatorname{supp} \boldsymbol{\ell}$, and $\alpha > 1 - \frac{1}{q}$.

(iii) All higher order chaoses scale slower than the leading chaos, i.e., if $c_{i,\boldsymbol{\ell}} \neq 0$ and $|\boldsymbol{\ell}| > q$, then $H(\boldsymbol{\ell}) < 1 - \frac{q}{2}(1-\alpha)$.

For the common α in Assumption 3.3, we define

$$H = 1 - \frac{q}{2}(1-\alpha) \in (1/2, 1). \tag{3.6}$$

Note that, if all active input coordinates, including those in higher chaoses, have the common exponent α, then Assumption 3.3(iii) follows immediately from Assumption 3.3(ii), since $1 - p(1-\alpha)/2 < H$ for $p > q$.

3.2 Main theorems

Recall the objects in (3.4). Define the bounded diagonal operator and its normalization by

$$Dv_i = d_i v_i, \qquad n^{-D} v_i = n^{-d_i} v_i \tag{3.7}$$

and the partial sums process and its continuous linear interpolation:

$$S_n(s) = T_n \sum_{t=1}^{\lfloor ns \rfloor} G[X_t], \qquad \widetilde{S}_n(s) = S_n(s) + (ns - \lfloor ns \rfloor) T_n G[X_{\lfloor ns \rfloor + 1}], \tag{3.8}$$

where $\{T_n\}$ is a suitable sequence of scaling operators. The random variable $\widetilde{S}_n$ is an element of $C([0,1]; \mathbb{H}_2)$. The following is our main result, identifying the scaling limits for (3.8).

Theorem 3.1. *Let $\{X_t : t \in \mathbb{Z}\}$ be a centered stationary Gaussian sequence with values in $\mathbb{H}_1$, and let $G \in L^2(\mathbb{H}_1, \gamma_Q; \mathbb{H}_2)$ be centered and nonzero, where γ_Q denotes the law of X_0. Suppose that Assumptions 3.1 and 3.2 hold. Let D be the bounded diagonal operator in (3.7), and denote by Z_D the $\mathbb{H}_2$-valued process given in (4.8) below. Then, as $n \to \infty$,*

$$n^{-D} \sum_{t=1}^{\lfloor ns \rfloor} G[X_t] \xrightarrow{f.d.d.} Z_D(s) \tag{3.9}$$

in $\mathbb{H}_2$, where $\xrightarrow{f.d.d.}$ denotes convergence in finite dimensional distribution. If, in addition, $d_* \doteq \inf_{i:\, \langle G, v_i \rangle_{\mathbb{H}_2} \neq 0} d_i > \frac{1}{2}$, then the process $\widetilde{S}_n$ in (3.8) with $T_n = n^{-D}$ converge weakly in $C([0,1]; \mathbb{H}_2)$ to a continuous version of Z_D.

The limit Z_D in (3.9) may contain several Hermite orders. If all surviving terms have order one, the limit is Gaussian.

We now consider the case where all input coordinates appearing in the leading chaos have a common memory exponent, which is a corollary of Theorem 3.1.

Corollary 3.1. *Let $\{X_t : t \in \mathbb{Z}\}$ be a centered stationary Gaussian sequence with values in $\mathbb{H}_1$, and let $G \in L^2(\mathbb{H}_1, \gamma_Q; \mathbb{H}_2)$ be centered and nonzero, where γ_Q denotes the law of X_0. Suppose that Assumptions 3.1 and 3.3 hold. Let H and $Z_H^{(q)}$ be given in (3.6) and (4.10), respectively. Then, as $n \to \infty$,*

$$n^{-H} \sum_{t=1}^{\lfloor ns \rfloor} G[X_t] \xrightarrow{f.d.d.} Z_H^{(q)}(s) \tag{3.10}$$

in $\mathbb{H}_2$ and, moreover, the process $\widetilde{S}_n$ in (3.8) with $T_n = n^{-H}$ converges weakly in $C([0,1]; \mathbb{H}_2)$ to a continuous version of $Z_H^{(q)}$.

3.3 Discussion on the assumptions

Remark 3.1. As pointed out in the Introduction, an interesting example where the double spectral diagonalization (3.1) holds is obtained when the spatial domain is a connected and compact two point homogeneous space $\mathbb{M}_d$ with topological dimension d. Invariance under the group of isometries leads to diagonalization of the covariance and spectral density operator families in the eigenfunctions of the Laplace–Beltrami operator on $\mathbb{M}_d$; see [40].

Remark 3.2. Replacing Assumption 3.1(iv) with the slightly stronger assumption $\sup_k \kappa_k < \infty$, we obtain a particularly convenient equivalent formulation of Assumption 3.1(iii). We can recast the spectral density operators as

$$\mathscr{F}_\omega = Q^{1/2} |\omega|^{-A} L_\omega \mathcal{K} Q^{1/2}, \qquad \text{for a.e. } 0 < |\omega| \leq \pi,$$

where the associated operators are diagonal and share a common eigenbasis $Q\phi_k = \lambda_k\phi_k$, define

$$A\phi_k = \alpha_k\phi_k, \qquad \mathcal{K}\phi_k = \kappa_k\phi_k, \qquad L_\omega\phi_k = L_k(|\omega|)\phi_k.$$

Then, we need that $\|A\|_{\mathcal{L}(\mathbb{H}_1)} < 1$, $\|\mathcal{K}\|_{\mathcal{L}(\mathbb{H}_1)} = \sup_k \kappa_k < \infty$, $0 \le L_\omega \le M\,\mathrm{Id}$ for a.e. $0 < |\omega| \le \pi$, and that $L_\omega h \to h$ for every $h \in \mathbb{H}_1$ as $\omega \to 0$.

The following remark gives some sufficient time-domain assumptions ensuring our spectral Assumption 3.1.

Remark 3.3. Assumption 3.1 holds, for example, if $Q\phi_k = \lambda_k\phi_k$, $\lambda_k > 0$, $\sum_k \lambda_k < \infty$, and the autocovariance operators are diagonalized by the same basis and satisfy

$$\Gamma(h)\phi_k = \lambda_k c_k h^{\alpha_k - 1} L_k(h)\phi_k, \qquad h \ge 1,$$

where $0 < \inf_k \alpha_k \le \alpha_k \le \sup_k \alpha_k < 1$, $0 < \inf_k c_k \le c_k \le \sup_k c_k < \infty$, and each L_k is measurable and slowly varying at infinity, with $\lim_{x\to\infty} L_k(x) = 1$. We additionally require

$$\sup_k \sum_{h=1}^{\infty} h^{\alpha_k - 1}|L_k(h) - 1| < \infty.$$

Indeed, the power-law cosine-series expansion, uniformly over the stated compact range of memory parameters, together with the weighted summability condition yields

$$\frac{f_k(\omega)}{\lambda_k} = \kappa_k|\omega|^{-\alpha_k} + R_k(\omega), \qquad \sup_k \|R_k\|_\infty < \infty, \qquad \kappa_k = \frac{c_k\Gamma(\alpha_k)\cos(\pi\alpha_k/2)}{\pi}.$$

The expression for κ_k is the standard scalar asymptotic constant; see [42, Proposition 2.2.14]. Since the κ_k's are uniformly bounded above and away from zero, and $\sup_k \|R_k\|_\infty < \infty$, Assumption 3.1 follows.

Remark 3.4. Suppose that G has hermite rank q. Then, for all $\ell \in \mathcal{L}_p, p \ge q$, we have that $H(\boldsymbol{\ell}) = 1 - \frac{1}{2}\sum_j \ell_j(1-\alpha_j) \le 1 - \frac{p}{2}(1-\alpha) \le 1 - \frac{q}{2}(1-\alpha)$, and the equality holds precisely when (i) $p = q$ and (ii) $\boldsymbol{\ell} \in \mathcal{L}_q$ and $\operatorname{supp}\boldsymbol{\ell} \subset D$.

4 Construction of the limit

In this section, we construct the limiting random variable Z_D in (3.9).

4.1 Spectral kernels

For $s \in [0,1]$, define the temporal kernel

$$\widehat{K}_s(y) = \int_0^s e^{ivy}\,dv = \begin{cases} \dfrac{e^{isy}-1}{iy}, & y \ne 0, \\ s, & y = 0. \end{cases} \tag{4.1}$$

Fix $\boldsymbol{\ell} \in \mathcal{L}_p$, $p \ge 1$, with $H(\boldsymbol{\ell}) > 1/2$, and let $k_1 \le \cdots \le k_p$ list the coordinate indices, with each j appearing exactly ℓ_j times. For $x = (x_1, \ldots, x_p)$, set

$$K_{s,\boldsymbol{\ell}}(x) = \widehat{K}_s\left(\sum_{a=1}^p x_a\right)\prod_{a=1}^p |x_a|^{-\alpha_{k_a}/2}, \tag{4.2}$$

$$L_{n,s,\boldsymbol{\ell}}(x) = n^{-H(\boldsymbol{\ell})-p/2}\sum_{t=1}^{\lfloor ns\rfloor} e^{it\sum_{a=1}^p x_a/n}\prod_{a=1}^p \left(\widetilde{f}_{k_a}(x_a/n)^{1/2}\mathbb{1}_{\{|x_a|\le\pi n\}}\right). \tag{4.3}$$

Define, moreover,

$$B_{\boldsymbol{\ell}} = \int_{\mathbb{R}^p} (1 \wedge |x_1 + \cdots + x_p|^{-1})^2 \prod_{a=1}^{p} \kappa_{k_a} |x_a|^{-\alpha_{k_a}} \, dx. \tag{4.4}$$

and, for $p \geq 2$, the convolution

$$\Psi_{\boldsymbol{\ell}}(y) = \int_{\mathbb{R}^{p-1}} \left(\prod_{a=1}^{p-1} \kappa_{k_a} |x_a|^{-\alpha_{k_a}} \right) \kappa_{k_p} \left| y - \sum_{a=1}^{p-1} x_a \right|^{-\alpha_{k_p}} dx_1 \cdots dx_{p-1}, \tag{4.5}$$

while, for $p = 1$, set $\Psi_{\boldsymbol{\ell}}(y) = \kappa_{k_1} |y|^{-\alpha_{k_1}}$.

Lemma 4.1. *For every $\boldsymbol{\ell} \in \mathcal{L}_p$, $p \geq 1$, satisfying $H(\boldsymbol{\ell}) > 1/2$, we have $0 < \Psi_{\boldsymbol{\ell}}(1) < \infty$ and*

$$\Psi_{\boldsymbol{\ell}}(y) = \Psi_{\boldsymbol{\ell}}(1) |y|^{1-2H(\boldsymbol{\ell})}, \qquad y \neq 0. \tag{4.6}$$

Moreover,

$$B_{\boldsymbol{\ell}} = \Psi_{\boldsymbol{\ell}}(1) \left(\frac{1}{1 - H(\boldsymbol{\ell})} + \frac{1}{H(\boldsymbol{\ell})} \right) < \infty. \tag{4.7}$$

Proof. For $p = 1$, we have $1 - 2H(\boldsymbol{\ell}) = -\alpha_{k_1}$ and $\Psi_{\boldsymbol{\ell}}(1) = \kappa_{k_1}$, so (4.6) follows directly from the definition of $\Psi_{\boldsymbol{\ell}}$.

For $p \geq 2$ and $1 \leq r \leq p$, set

$$\eta_r = \sum_{a=1}^{r} \alpha_{k_a} - r + 1 = 1 - \sum_{a=1}^{r} (1 - \alpha_{k_a}),$$

Since the terms $1 - \alpha_{k_a}$ are all positive and their total is $\beta(\boldsymbol{\ell}) < 1$, $\eta_r \in (0, 1)$ for all $1 \leq r \leq p$.

We use now the form of Ψ in (4.5) to prove (4.6). Note that it suffices to prove that $\Psi_\ell(y) = C\Psi_\ell(1)|y|^{1-2H(\ell)}$, since then we can obtain the constant by evaluating Ψ_ℓ at one.

We prove that, for every $r = 1, \ldots, p$, there exists $0 < C_r < \infty$ such that, whenever $y \neq 0$,

$$\int_{\mathbb{R}^{r-1}} \left(\prod_{a=1}^{r-1} \kappa_{k_a} |x_a|^{-\alpha_{k_a}} \right) \kappa_{k_r} \left| y - \sum_{a=1}^{r-1} x_a \right|^{-\alpha_{k_r}} dx_1 \cdots dx_{r-1} = C_r |y|^{-\eta_r}.$$

For $r = 1$, the left-hand side is understood as $\kappa_{k_1} |y|^{-\alpha_{k_1}}$. Since $\eta_1 = \alpha_{k_1}$, the identity holds with $C_1 = \kappa_{k_1}$.

Now fix $2 \leq r \leq p$ and assume the identity holds for $r - 1$. By Tonelli's theorem and the induction hypothesis,

$$\begin{aligned} \int_{\mathbb{R}^{r-1}} \left(\prod_{a=1}^{r-1} \kappa_{k_a} |x_a|^{-\alpha_{k_a}} \right) \kappa_{k_r} \left| y - \sum_{a=1}^{r-1} x_a \right|^{-\alpha_{k_r}} dx_1 \cdots dx_r &= C_{r-1} \kappa_{k_r} \int_{\mathbb{R}} |u|^{-\eta_{r-1}} |y - u|^{-\alpha_{k_r}} \, du \\ &= C_{r-1} \kappa_{k_r} |y|^{-\eta_r} \int_{\mathbb{R}} |v|^{-\eta_{r-1}} |1 - v|^{-\alpha_{k_r}} \, dv = C_r |y|^{-\eta_r}, \end{aligned}$$

where for the third equality we used the substitution $u = yv$ and that $\eta_r = \eta_{r-1} + \alpha_{k_r} - 1$; for the fourth equality we have set $C_r = C_{r-1} \kappa_{k_r} \int_{\mathbb{R}} |v|^{-\eta_{r-1}} |1 - v|^{-\alpha_{k_r}} \, dv \in (0, \infty)$, where notice that the finiteness of the integral near zero and one is due to $0 < \eta_{r-1}, \alpha_{k_r} < 1$, and at infinity because $\eta_{r-1} + \alpha_{k_r} = 1 + \eta_r > 1$. This proves the induction step.

To obtain $B_{\boldsymbol{\ell}}$, replace x_p by $y = x_1 + \cdots + x_p$. Then, using Tonelli's theorem and the definition of $\Psi_{\boldsymbol{\ell}}$,

$$B_{\boldsymbol{\ell}} = \int_{\mathbb{R}} (1 \wedge |y|^{-1})^2 \Psi_{\boldsymbol{\ell}}(y) \, dy = 2\Psi_{\boldsymbol{\ell}}(1) \left(\int_0^1 y^{1-2H(\boldsymbol{\ell})} \, dy + \int_1^\infty y^{-1-2H(\boldsymbol{\ell})} \, dy \right).$$

Both integrals are finite because $1/2 < H(\boldsymbol{\ell}) < 1$, proving (4.7). □

4.2 Definition of the limit

4.2.1 The general case

On the common spectral probability space introduced in Section 2.3, define, with $\kappa^{\ell} = \prod_j \kappa_j^{\ell_j}$,

$$Z_D(s) = \sum_{i=1}^{\infty} Z_i(s) v_i \quad \text{with} \quad Z_i(s) = \sum_{\boldsymbol{\ell} \in \mathcal{D}_i} c_{i,\boldsymbol{\ell}} \kappa^{\boldsymbol{\ell}/2} I_{|\boldsymbol{\ell}|}(J_{\boldsymbol{\ell}} K_{s,\boldsymbol{\ell}}), \tag{4.8}$$

or, equivalently, groupping the terms of order p into a kernel $h_{p,s} \in \mathfrak{H}^{\odot p} \otimes \mathbb{H}_2$, we can write

$$Z_D(s) = \sum_{p \geq q} \mathcal{I}_p(h_{p,s}), \qquad h_{p,s} \doteq \sum_{i=1}^{\infty} \sum_{\ell \in \mathcal{D}_i, |\boldsymbol{\ell}| = p} c_{i,\boldsymbol{\ell}} \kappa^{\boldsymbol{\ell}/2} J_{\boldsymbol{\ell}} K_{s,\boldsymbol{\ell}} \otimes v_i. \tag{4.9}$$

By using (2.8), the fact that label supports for different multiplicity indices are disjoint, and the kernel estimates (5.1) below, we can infer that

$$\begin{aligned} \mathrm{E}\, \|Z_D(s)\|_{\mathbb{H}_2}^2 &= \sum_{i=1}^{\infty} \sum_{\boldsymbol{\ell} \in \mathcal{D}_i} \boldsymbol{\ell}!\, |c_{i,\boldsymbol{\ell}}|^2 \|\kappa^{\boldsymbol{\ell}/2} K_{s,\boldsymbol{\ell}}\|_{L^2(\mathbb{R}^{|\boldsymbol{\ell}|})}^2 \\ &\leq \sum_{i=1}^{\infty} \sum_{\boldsymbol{\ell} \in \mathcal{D}_i} \boldsymbol{\ell}!\, |c_{i,\boldsymbol{\ell}}|^2 C/(2d_i - 1) \leq C \sum_{i=1}^{\infty} \frac{\mathrm{E}\, |\langle G[X_0], v_i \rangle_{\mathbb{H}_2}|^2}{2d_i - 1} < \infty, \end{aligned}$$

from Assumption 3.2(ii).

4.2.2 The common exponent case

Let q and α be as in Assumption 3.3. Define

$$K_s(x_1, \ldots, x_q) := \widehat{K}_s\left(\sum_{a=1}^{q} x_a\right) \prod_{a=1}^{q} |x_a|^{-\alpha/2}, \qquad (x_1, \ldots, x_q) \in \mathbb{R}^q,$$

where $\widehat{K}_s$ is given by (4.1). By Assumption 3.3(ii) and (4.2), we have $K_{s,\ell} = K_s$ for every $\ell \in \mathcal{L}_q$ such that $c_{i,\ell} \neq 0$ for some i. We then define

$$Z_H^{(q)}(s) = \sum_{i=1}^{\infty} I_q(\mathcal{K}_{i,s}) v_i = \mathcal{I}_q\left(\sum_{i=1}^{\infty} \mathcal{K}_{i,s} \otimes v_i\right), \quad \mathcal{K}_{i,s} = \sum_{\boldsymbol{\ell} \in \mathcal{L}_q} c_{i,\boldsymbol{\ell}} \kappa^{\boldsymbol{\ell}/2} J_{\boldsymbol{\ell}} K_s \quad \text{in } \mathfrak{H}^{\odot q}. \tag{4.10}$$

where, recall (4.2). Note that when $s > 0$, $K_s \in L^2(\mathbb{R}^q)$ if and only if $1 - 1/q < \alpha < 1$. The isometries (2.5)–(2.8) then imply

$$\mathrm{E}\, \|Z_H^{(q)}(s)\|_{\mathbb{H}_2}^2 = \|K_s\|_{L^2(\mathbb{R}^q)}^2 \sum_{i=1}^{\infty} \sum_{\boldsymbol{\ell} \in \mathcal{L}_q} \boldsymbol{\ell}!\, |c_{i,\boldsymbol{\ell}}|^2 \kappa^{\boldsymbol{\ell}} < \infty.$$

5 Proofs of the limit results

We derive convergence of the kernel functions that correspond to pre- and postlimits and prove a reduction principle. Finally, we combine these two results to prove Theorem 3.1, and conclude with the proof of Corollary 3.1.

5.1 Kernel convergence

Recall the prelimit kernels $L_{n,s,\boldsymbol{\ell}}$ from (4.3) and the limit kernels $K_{s,\boldsymbol{\ell}}$ from (4.2). The following lemma states a convergence result for these kernels.

Proposition 5.1. *Suppose Assumptions 3.1(ii)–(iv). For every $\boldsymbol{\ell} \in \mathcal{L}_p$ supported on active coordinates with $H(\boldsymbol{\ell}) > 1/2$, and every $s \in [0,1]$, there is a constant C, independent of the index and its order, such that*

$$\sup_{n\geq 1}\sup_{s\in[0,1]} \|L_{n,s,\boldsymbol{\ell}}\|^2_{L^2(\mathbb{R}^p)} \leq \frac{C}{2H(\boldsymbol{\ell})-1}, \qquad \sup_{s\in[0,1]} \|\kappa^{\boldsymbol{\ell}/2} K_{s,\boldsymbol{\ell}}\|^2_{L^2(\mathbb{R}^p)} \leq \frac{C}{2H(\boldsymbol{\ell})-1}, \tag{5.1}$$

Suppose, in addition, Assumptions 3.2(i),(ii). Then, for the dominant sets $\mathcal{D}_i$ and any fixed $s_1,\dots,s_m \in [0,1]$, $\theta_1,\dots,\theta_m \in \mathbb{R}$,

$$\sum_{i=1}^{\infty}\sum_{\boldsymbol{\ell}\in\mathcal{D}_i} \boldsymbol{\ell}!\,|c_{i,\boldsymbol{\ell}}|^2 \left\| \sum_{a=1}^{m} \theta_a (L_{n,s_a,\boldsymbol{\ell}} - \kappa^{\boldsymbol{\ell}/2} K_{s_a,\boldsymbol{\ell}}) \right\|^2_{L^2(\mathbb{R}^{|\boldsymbol{\ell}|})} \to 0. \tag{5.2}$$

Proof. We first establish convergence in $L^2(\mathbb{R}^p)$ of the kernels $L_{n,s,\boldsymbol{\ell}}$ to $\kappa^{\boldsymbol{\ell}/2}K_{s,\boldsymbol{\ell}}$. This convergence is then used to infer (5.1) and (5.2).

For the $L^2(\mathbb{R}^p)$-convergence, we first establish almost-everywhere convergence and verify then that dominated convergence theorem may be applied by showing integrability of the respective objects. Fix $\boldsymbol{\ell} \in \mathcal{L}_p$ with $H(\boldsymbol{\ell}) > 1/2$, write $\mathbf{k} = \mathbf{k}(\boldsymbol{\ell})$, and

$$\widehat{K}_{n,s}(y) \doteq \frac{1}{n}\sum_{t=1}^{\lfloor ns\rfloor} e^{ity/n}, \qquad y = x_1 + \cdots + x_p. \tag{5.3}$$

Fix also $s \in (0,1]$; the case $s = 0$ is immediate because both kernels vanish. Using the definition in (4.3), we have, with further explanations given below,

$$\begin{aligned} L_{n,s,\boldsymbol{\ell}}(x) &= n^{-H(\boldsymbol{\ell})-p/2} \sum_{t=1}^{\lfloor ns\rfloor} e^{it\sum_{a=1}^p x_a/n} \prod_{a=1}^{p} \widetilde{f}_{k_a}(x_a/n)^{1/2} \mathbf{1}_{\{|x_a|\leq \pi n\}} \\ &= \widehat{K}_{n,s}(y) \prod_{a=1}^{p} \left[n^{-\alpha_{k_a}/2} \widetilde{f}_{k_a}(x_a/n)^{1/2} \mathbf{1}_{\{|x_a|\leq\pi n\}} \right] \qquad (5.4) \\ &\to \widehat{K}_s(y) \prod_{a=1}^{p} \kappa_{k_a}^{1/2} |x_a|^{-\alpha_{k_a}/2} = \kappa^{\boldsymbol{\ell}/2} K_{s,\boldsymbol{\ell}}(x) \quad \text{for a.e. } x \in \mathbb{R}^p, \qquad (5.5) \end{aligned}$$

where (5.4) follows by (5.3) and since $-1-\frac{1}{2}\sum_{a=1}^p \alpha_{k_a} = -H(\boldsymbol{\ell})-p/2$. For (5.5), the convergence of each spectral factor follows from Assumption 3.1(iii), since $x_a/n \to 0$ and, for fixed $x_a \neq 0$, the indicator equals one for all sufficiently large n. The convergence $\widehat{K}_{n,s}(y) \to \widehat{K}_s(y)$, with $\widehat{K}_s$ defined in (4.1), follows from the end of p. 289, (5.3.10), (5.3.13), and p. 291 of [42]. The final equality uses $\kappa^{\boldsymbol{\ell}/2} \doteq \prod_{a=1}^p \kappa_{k_a}^{1/2}$ and the definition of $K_{s,\boldsymbol{\ell}}$ in (4.2).

For integrability, we introduce the set $A_n = \{x \in \mathbb{R}^p : |x_1 + \cdots + x_p| \leq \pi n\}$ and consider both A_n and its complement separately. On A_n, we get

$$\begin{aligned} |\widehat{K}_{n,s}(y)| &\leq \min\left\{ \frac{\lfloor ns\rfloor}{n}, \frac{1}{n|\sin(y/(2n))|} \right\} \leq \min\left\{ 1, \frac{1}{n|\sin(y/(2n))|} \right\}, \\ &\leq \min\{1, \pi|y|^{-1}\} \leq \pi(1 \wedge |y|^{-1}), \qquad |y| \leq \pi n, \end{aligned} \tag{5.6}$$

where, in the first line, the two bounds follow respectively from the triangle inequality and (A.2.2) in [42]; we then use $\lfloor ns\rfloor/n \le 1$. On $|y| \le \pi n$, the inequality $|\sin(y/(2n))| \ge |y|/(\pi n)$ gives the second line. Then, using (5.4),

$$|L_{n,s,\boldsymbol{\ell}}(x)\mathbf{1}_{A_n}(x)|^2 = \left|\widehat{K}_{n,s}(y)\prod_{a=1}^{p}\left[n^{-\alpha_{k_a}/2}\widetilde{f}_{k_a}(x_a/n)^{1/2}\mathbf{1}_{\{|x_a|\le\pi n\}}\right]\right|^2 \mathbf{1}_{A_n}(x)$$

$$\le M^p\kappa^{\boldsymbol{\ell}}|\widehat{K}_{n,s}(y)|^2\prod_{a=1}^{p}|x_a|^{-\alpha_{k_a}}\mathbf{1}_{A_n}(x) \tag{5.7}$$

$$\le \pi^2 M^p\kappa^{\boldsymbol{\ell}}(1\wedge|y|^{-1})^2\prod_{a=1}^{p}|x_a|^{-\alpha_{k_a}} \doteq g_{\boldsymbol{\ell}}(x), \tag{5.8}$$

where (5.7) follows by Assumption 3.1(iii) and $\prod_{a=1}^{p}\kappa_{k_a} = \kappa^{\boldsymbol{\ell}}$; (5.8) is due to (5.6). Combining this bound with (5.6) on A_n gives the last line. It remains to show that $g_{\boldsymbol{\ell}}(x)$ is integrable, that is,

$$\int_{\mathbb{R}^p} g_{\boldsymbol{\ell}}(x)\,dx = \pi^2 M^p\int_{\mathbb{R}^p}(1\wedge|x_1+\cdots+x_p|^{-1})^2\prod_{a=1}^{p}\kappa_{k_a}|x_a|^{-\alpha_{k_a}}\,dx = \pi^2 M^p B_{\boldsymbol{\ell}} < \infty,$$

with $B_{\boldsymbol{\ell}}$ as in (4.4). Finiteness was established in (4.7). By (5.5), for almost every $x\in\mathbb{R}^p$,

$$L_{n,s,\boldsymbol{\ell}}(x)\mathbf{1}_{A_n}(x) \to \kappa^{\boldsymbol{\ell}/2}K_{s,\boldsymbol{\ell}}(x),$$

using $\mathbf{1}_{A_n}(x)\to 1$ that holds since every fixed x belongs to A_n for all sufficiently large n. Finally,

$$\left|L_{n,s,\boldsymbol{\ell}}(x)\mathbf{1}_{A_n}(x) - \kappa^{\boldsymbol{\ell}/2}K_{s,\boldsymbol{\ell}}(x)\right|^2 \le 2|L_{n,s,\boldsymbol{\ell}}(x)\mathbf{1}_{A_n}(x)|^2 + 2|\kappa^{\boldsymbol{\ell}/2}K_{s,\boldsymbol{\ell}}(x)|^2 \le 4g_{\boldsymbol{\ell}}(x),$$

using $|u-v|^2\le 2|u|^2+2|v|^2$ and since by (5.5), we have $|\kappa^{\boldsymbol{\ell}/2}K_{s,\boldsymbol{\ell}}(x)|^2 = \lim_{n\to\infty}|L_{n,s,\boldsymbol{\ell}}(x)\mathbf{1}_{A_n}(x)|^2 \le g_{\boldsymbol{\ell}}(x)$. The integrand in the last line converges to zero almost everywhere and is bounded by the integrable function $4g_{\boldsymbol{\ell}}$.

We can now apply dominated convergence on A_n to prove (5.9)

$$\left\|L_{n,s,\boldsymbol{\ell}}\mathbf{1}_{A_n} - \kappa^{\boldsymbol{\ell}/2}K_{s,\boldsymbol{\ell}}\right\|^2_{L^2(\mathbb{R}^p)} = \int_{\mathbb{R}^p}\left|L_{n,s,\boldsymbol{\ell}}(x)\mathbf{1}_{A_n}(x) - \kappa^{\boldsymbol{\ell}/2}K_{s,\boldsymbol{\ell}}(x)\right|^2 dx \to 0. \tag{5.9}$$

It remains to control the prelimit kernel on A_n^c. On A_n^c, use the integral of $|\widehat{K}_{n,s}|^2$. Parseval's identity in (A.1.4) of [42] gives

$$\int_{-\pi n}^{\pi n}|\widehat{K}_{n,s}(y)|^2\,dy = \frac{1}{n}\int_{-\pi}^{\pi}\left|\sum_{t=1}^{\lfloor ns\rfloor}e^{itx}\right|^2 dx = \frac{2\pi\lfloor ns\rfloor}{n} \le 2\pi. \tag{5.10}$$

The integrand is $2\pi n$-periodic. Since p is an integer, integration over any interval of length $2p\pi n$ therefore gives at most $2\pi p$.

The support of $L_{n,s,\boldsymbol{\ell}}$ is contained in $[-\pi n,\pi n]^p$, so $|y|\le p\pi n$ on that support. Then, with further explanations given below,

$$\|L_{n,s,\boldsymbol{\ell}}\mathbf{1}_{A_n^c}\|^2_{L^2(\mathbb{R}^p)} \le M^{|\boldsymbol{\ell}|}\int_{\substack{|x_a|\le\pi n,\ a=1,\dots,p\\ |x_1+\cdots+x_p|>\pi n}}|\widehat{K}_{n,s}(x_1+\cdots+x_p)|^2\prod_{a=1}^{p}\kappa_{k_a}|x_a|^{-\alpha_{k_a}}\,dx.$$

$$\le M^{|\boldsymbol{\ell}|}\int_{\pi n<|y|\le p\pi n}|\widehat{K}_{n,s}(y)|^2\Psi_{\boldsymbol{\ell}}(y)\,dy \tag{5.11}$$

$$\le M^{|\boldsymbol{\ell}|}\Psi_{\boldsymbol{\ell}}(1)(\pi n)^{1-2H(\boldsymbol{\ell})}\int_{-p\pi n}^{p\pi n}|\widehat{K}_{n,s}(y)|^2\,dy \tag{5.12}$$

$$\le \pi p M^{|\boldsymbol{\ell}|}B_{\boldsymbol{\ell}}n^{1-2H(\boldsymbol{\ell})} \to 0. \tag{5.13}$$

For the first inequality, we used (5.4) and Assumption 3.1(iii). The inequality (5.11) is due to substituting $x_p = y - \sum_{a=1}^{p-1} x_a$, and using the definition of $\Psi_{\boldsymbol{\ell}}$ in (4.5). For (5.12), we use (4.6) of Lemma 4.1 and $|y| > \pi n$. The last inequality follows from (5.10), $\Psi_{\boldsymbol{\ell}}(1) \le B_{\boldsymbol{\ell}}/2$, and $\pi^{1-2H(\boldsymbol{\ell})} \le 1$. The bound is uniform in $s \in [0,1]$ and tends to zero because $2H(\boldsymbol{\ell}) - 1 > 0$. Combining (5.9) and (5.13), the triangle inequality gives

$$L_{n,s,\boldsymbol{\ell}} \to \kappa^{\boldsymbol{\ell}/2} K_{s,\boldsymbol{\ell}} \quad \text{in } L^2(\mathbb{R}^p). \tag{5.14}$$

Proof of (5.1)*:* By the isometry (2.8), the coordinate spectral representation under Assumption 3.1(ii) and the change of variables $x_a = n\omega_a$,

$$\|L_{n,s,\boldsymbol{\ell}}\|^2_{L^2(\mathbb{R}^p)} = \frac{n^{-2H(\boldsymbol{\ell})}}{\boldsymbol{\ell}!} \operatorname{Var}\left(\sum_{t=1}^{\lfloor ns \rfloor} H_{\boldsymbol{\ell}}[X_t]\right) \le \frac{C}{2H(\boldsymbol{\ell})-1}\left(\frac{\lfloor ns \rfloor}{n}\right)^{2H(\boldsymbol{\ell})} \le \frac{C}{2H(\boldsymbol{\ell})-1}.$$

Here (6.4) is applied with $d = H(\boldsymbol{\ell})$. Fatou's lemma and the convergence in (5.14) give

$$\|\kappa^{\boldsymbol{\ell}/2} K_{s,\boldsymbol{\ell}}\|^2_{L^2(\mathbb{R}^p)} \le \liminf_{n\to\infty} \|L_{n,s,\boldsymbol{\ell}}\|^2_{L^2(\mathbb{R}^p)} \le \frac{C}{2H(\boldsymbol{\ell})-1}.$$

Both estimates hold for every $s \in [0,1]$, proving (5.1). The variance lemma was proved directly from the spectral densities and does not use kernel convergence or any coefficient summability.

Proof of (5.2)*:* Fix $s_1, \dots, s_m \in [0,1]$ and $\theta_1, \dots, \theta_m \in \mathbb{R}$. For $\boldsymbol{\ell} \in \mathcal{D}_i$, we have $H(\boldsymbol{\ell}) = d_i$, with $d_i \in (1/2, 1)$ by Assumption 3.2(i). Then, with further explanations given below,

$$\begin{aligned}
&\sum_{i=1}^{\infty} \sum_{\boldsymbol{\ell}\in\mathcal{D}_i} \boldsymbol{\ell}! \, |c_{i,\boldsymbol{\ell}}|^2 \left\| \sum_{a=1}^{m} \theta_a (L_{n,s_a,\boldsymbol{\ell}} - \kappa^{\boldsymbol{\ell}/2} K_{s_a,\boldsymbol{\ell}}) \right\|^2_{L^2(\mathbb{R}^{|\boldsymbol{\ell}|})} \\
&\le \sum_{i=1}^{\infty} \sum_{\boldsymbol{\ell}\in\mathcal{D}_i} \boldsymbol{\ell}! \, |c_{i,\boldsymbol{\ell}}|^2 \frac{4C \left(\sum_{a=1}^{m} |\theta_a|\right)^2}{2d_i - 1} \qquad (5.15) \\
&\le 4C \left(\sum_{a=1}^{m} |\theta_a|\right)^2 \sum_{i=1}^{\infty} \frac{\mathrm{E}\,|\langle G[X_0], v_i\rangle_{\mathbb{H}_2}|^2}{2d_i - 1} < \infty, \qquad (5.16)
\end{aligned}$$

where (5.15) is due to triangle inequality and (5.1), (5.16) follows by Parseval's identity and Assumption 3.2(ii). For each fixed index the squared error tends to zero by (5.14). Dominated convergence of the countable series proves (5.2). □

5.2 Reduction Principle

For the following proofs, we first introduce the required notation. Recall $\mathcal{L}$ from (2.2). Let $\mathcal{Q} = (\mathcal{Q}_i)_{i\ge1}$, where each $\mathcal{Q}_i \subseteq \mathcal{L} \setminus \{0\}$ consists of indices supported on active coordinates. Fix $d_i \in (1/2, 1)$ and recall the operator D from (3.7). For $N \ge 1$, define

$$\mathcal{F}_N(\mathcal{Q}) \doteq \{(i,\boldsymbol{\ell}) : 1 \le i \le N, \quad \boldsymbol{\ell} \in \mathcal{Q}_i, \quad |\boldsymbol{\ell}| \le N, \quad \ell_j = 0 \text{ for all } j > N\},$$

and write

$$G_{\mathcal{Q}} = \sum_{i=1}^{\infty} \sum_{\boldsymbol{\ell}\in\mathcal{Q}_i} c_{i,\boldsymbol{\ell}} H_{\boldsymbol{\ell}} v_i, \qquad G_{\mathcal{Q}}^{(N)} = \sum_{(i,\boldsymbol{\ell})\in\mathcal{F}_N(\mathcal{Q})} c_{i,\boldsymbol{\ell}} H_{\boldsymbol{\ell}} v_i. \tag{5.17}$$

The sets $\mathcal{F}_N(\mathcal{Q})$ are finite and increase to all indices in $\mathcal{Q}$. By Hermite orthogonality and Parseval's identity, $G_{\mathcal{Q}}^{(N)}$ is an orthogonal projection of $G_{\mathcal{Q}}$ onto a finite-dimensional subspace of $L^2(\mathbb{H}_1, \gamma_Q; \mathbb{H}_2)$ and converges to $G_{\mathcal{Q}}$ in this space. Each projection takes values in $\operatorname{span}\{v_1, \dots, v_N\}$ and depends only on the first N standardized Gaussian coordinates.

Proposition 5.2. *Recall $G_{\mathcal{D}}$ from (3.5). Suppose Assumptions 3.1(ii)–(iv), 3.2(i),(ii). Then, as $n \to \infty$,*

$$\sup_{s\in[0,1]} \mathrm{E}\left\| n^{-D} \sum_{t=1}^{\lfloor ns \rfloor} (G[X_t] - G_{\mathcal{D}}[X_t]) \right\|_{\mathbb{H}_2}^2 \to 0.$$

Proof. We control each nondominant term uniformly over partial-sum lengths. Recall $\mathcal{N}_i$ from (3.4), i.e., the set of indices that corresponds to nondominat terms. Fix i and $\boldsymbol{\ell} \in \mathcal{N}_i$, and write $p = |\boldsymbol{\ell}|$ and $\beta = \beta(\boldsymbol{\ell})$. Under Assumptions 3.1(ii)–(iv), applying (6.3) of Lemma 6.2 gives

$$\max_{0\le m\le n} n^{-2d_i} \operatorname{Var}\left(\sum_{t=1}^{m} H_{\boldsymbol{\ell}}[X_t] \right) \le \boldsymbol{\ell}! C^p \begin{cases} \left(1 + \frac{2}{1-\beta}\right) n^{-2(d_i - H(\boldsymbol{\ell}))}, & \beta \in (0,1), \\ n^{1-2d_i}(1 + 2\log n), & \beta = 1, \\ \left(1 + \frac{2}{\beta - 1}\right) n^{1-2d_i}, & \beta > 1, \end{cases} \to 0. \quad (5.18)$$

For (5.18), the variance bounds are increasing functions of $m \ge 1$, so we may replace m by n in their upper bounds; the case $m = 0$ is immediate. When $0 < \beta < 1$, we use $2 - \beta = 2H(\boldsymbol{\ell})$. Moreover, $\boldsymbol{\ell} \in \mathcal{N}_i$ and $H(\boldsymbol{\ell}) > 1/2$ imply $H(\boldsymbol{\ell}) < d_i$, so $n^{-2(d_i - H(\boldsymbol{\ell}))} \to 0$. When $\beta = 1$, the condition $d_i > 1/2$ in Assumption 3.2(i) implies that $n^{1-2d_i}(1+2\log n) \to 0$. The same condition gives $n^{1-2d_i} \to 0$ when $\beta > 1$.

Set $F = G - G_{\mathcal{D}} = G_{\mathcal{N}}$ and $F^{(N)} = G_{\mathcal{N}}^{(N)}$, using the representation in (5.17) with $\mathcal{N} = (\mathcal{N}_i)_{i\ge 1}$. For each fixed N, Lemma 2.1 gives

$$\sup_{s\in[0,1]} \mathrm{E}\left\| n^{-D} \sum_{t=1}^{\lfloor ns \rfloor} F^{(N)}[X_t] \right\|_{\mathbb{H}_2}^2 \le \sum_{(i,\boldsymbol{\ell})\in\mathcal{F}_N(\mathcal{N})} |c_{i,\boldsymbol{\ell}}|^2 \max_{0\le m\le n} n^{-2d_i} \operatorname{Var}\left(\sum_{t=1}^{m} H_{\boldsymbol{\ell}}[X_t] \right) \to 0 \quad (5.19)$$

The sum is finite and each term tends to zero by (5.18). Recall that $H(\boldsymbol{\ell}) \le d_i$ for every $\boldsymbol{\ell} \in \mathcal{N}_i$, so Lemma 6.3 applies with $\mathcal{Q} = \mathcal{N}$. Combining its bound with (5.19) and the decomposition $F = F^{(N)} + (F - F^{(N)})$, we obtain, for every $N \ge 1$,

$$\begin{aligned} \limsup_{n\to\infty} \sup_{s\in[0,1]} \mathrm{E}\left\| n^{-D} \sum_{t=1}^{\lfloor ns \rfloor} F[X_t] \right\|_{\mathbb{H}_2}^2 &\le 2 \limsup_{n\to\infty} \sup_{s\in[0,1]} \mathrm{E}\left\| n^{-D} \sum_{t=1}^{\lfloor ns \rfloor} F^{(N)}[X_t] \right\|_{\mathbb{H}_2}^2 \\ &\quad + 2 \limsup_{n\to\infty} \sup_{s\in[0,1]} \mathrm{E}\left\| n^{-D} \sum_{t=1}^{\lfloor ns \rfloor} (F - F^{(N)})[X_t] \right\|_{\mathbb{H}_2}^2 \\ &\le 2C \sum_{\substack{i\ge 1,\ \boldsymbol{\ell}\in\mathcal{N}_i \\ (i,\boldsymbol{\ell})\notin\mathcal{F}_N(\mathcal{N})}} \frac{\boldsymbol{\ell}! |c_{i,\boldsymbol{\ell}}|^2}{2d_i - 1} \\ &= 2C \sum_{i=1}^{\infty} \sum_{\boldsymbol{\ell}\in\mathcal{N}_i} \frac{\boldsymbol{\ell}! |c_{i,\boldsymbol{\ell}}|^2}{2d_i - 1} - 2C \sum_{(i,\boldsymbol{\ell})\in\mathcal{F}_N(\mathcal{N})} \frac{\boldsymbol{\ell}! |c_{i,\boldsymbol{\ell}}|^2}{2d_i - 1} \to 0, \end{aligned}$$

as $N \to \infty$. The first inequality follows from $\|u + v\|_{\mathbb{H}_2}^2 \le 2\|u\|_{\mathbb{H}_2}^2 + 2\|v\|_{\mathbb{H}_2}^2$. For the second inequality, the first term vanishes by (5.19), while the second is bounded by Lemma 6.3, uniformly in n and s. The convergence follows, since, by Parseval's identity and Assumption 3.2(ii), we have

$$\sum_{i=1}^{\infty} \sum_{\boldsymbol{\ell}\in\mathcal{N}_i} \frac{\boldsymbol{\ell}! |c_{i,\boldsymbol{\ell}}|^2}{2d_i - 1} \le \sum_{i=1}^{\infty} \frac{1}{2d_i - 1} \sum_{\boldsymbol{\ell}\in\mathcal{L}} \boldsymbol{\ell}! |c_{i,\boldsymbol{\ell}}|^2 = \sum_{i=1}^{\infty} \frac{\mathrm{E}\,|\langle G[X_0], v_i \rangle_{\mathbb{H}_2}|^2}{2d_i - 1} < \infty.$$

□

5.3 Proof of Theorem 3.1

Proof. We combine Propositions 5.1 and 5.2. We first prove convergence of finite dimensional distributions in a product space of $\mathbb{H}_2$. We then consider the continuous interpolations (3.8) in $C([0,1];\mathbb{H}_2)$ and tightness.

Convergence of finite dimensional distributions: Recall $G_{\mathcal{D}}$ from (3.5), and write

$$A_n(s) = n^{-D}\sum_{t=1}^{\lfloor ns\rfloor} G_{\mathcal{D}}[X_t], \qquad R_n(s) = n^{-D}\sum_{t=1}^{\lfloor ns\rfloor}(G[X_t] - G_{\mathcal{D}}[X_t]). \tag{5.20}$$

It then suffices to prove convergence in distribution of $A_n(s)$ and that $R_n(s)$ is negligible to infer convergence of $A_n + R_n$ matching the prelimit in (3.9).

For $N \geq 1$, define the finite projections, using the notation (5.17) with $\mathcal{D} = (\mathcal{D}_i)_{i\geq 1}$,

$$A_n^{(N)}(s) = n^{-D}\sum_{t=1}^{\lfloor ns\rfloor} G_{\mathcal{D}}^{(N)}[X_t], \qquad Z_D^{(N)}(s) = \sum_{(i,\boldsymbol{\ell})\in\mathcal{F}_N(\mathcal{D})} c_{i,\boldsymbol{\ell}}\kappa^{\boldsymbol{\ell}/2} I_{|\boldsymbol{\ell}|}(J_{\boldsymbol{\ell}}K_{s,\boldsymbol{\ell}})v_i.$$

We first prove convergence for fixed N. For every coordinate spectral measure $\widehat{W}_j$, define

$$\widehat{W}_{j,n}(B) = n^{1/2}\widehat{W}_j(B/n), \qquad B \subset \mathbb{R} \text{ Borel with finite Lebesgue measure.}$$

The rescaled measures have Lebesgue control measure, since $n|B/n| = |B|$, and retain Hermitian symmetry and independence across coordinate labels. Hence the finitely many rescaled measures have the same joint law as the original measures. Applying the change of variables $x_a = n\omega_a$ simultaneously to the finite collection of multiple integrals gives

$$A_n^{(N)}(s) \overset{\text{f.d.d.}}{=} \widehat{A}_n^{(N)}(s) \doteq \sum_{(i,\boldsymbol{\ell})\in\mathcal{F}_N(\mathcal{D})} c_{i,\boldsymbol{\ell}} I_{|\boldsymbol{\ell}|}(J_{\boldsymbol{\ell}}L_{n,s,\boldsymbol{\ell}})v_i. \tag{5.21}$$

Here we use (2.9) and $H(\boldsymbol{\ell}) = d_i$ on $\mathcal{D}_i$. This is the usual joint change-of-measure property of multiple Wiener integrals; see [42, Appendix B.2, equation (B.2.15)]. The same rescaling is used for all selected times, labels and chaos orders, so (5.21) is a joint equality in law.

Fix $s_1, \dots, s_m \in [0,1]$. The labelled isometry and orthogonality give

$$\mathrm{E}\big\|(\widehat{A}_n^{(N)}(s_a) - Z_D^{(N)}(s_a))_{a=1}^m\big\|_{\mathbb{H}_2^m}^2 = \sum_{a=1}^m \sum_{(i,\boldsymbol{\ell})\in\mathcal{F}_N(\mathcal{D})} \boldsymbol{\ell}!|c_{i,\boldsymbol{\ell}}|^2 \|L_{n,s_a,\boldsymbol{\ell}} - \kappa^{\boldsymbol{\ell}/2}K_{s_a,\boldsymbol{\ell}}\|_{L^2}^2 \to 0.$$

There are only finitely many summands, and each tends to zero by (5.14). Thus, for fixed N, $(A_n^{(N)}(s_a))_{a=1}^m$ converges in distribution to $(Z_D^{(N)}(s_a))_{a=1}^m$ in the finite-dimensional subspace $\mathrm{span}\{v_1,\dots,v_N\}^m$.

It remains to remove the projection. By Lemma 6.3 and (5.1), respectively,

$$\sup_{n\geq 1}\sup_{s\in[0,1]} \mathrm{E}\,\|A_n(s) - A_n^{(N)}(s)\|_{\mathbb{H}_2}^2 \leq C \sum_{\substack{i\geq 1,\ \boldsymbol{\ell}\in\mathcal{D}_i\\ (i,\boldsymbol{\ell})\notin\mathcal{F}_N(\mathcal{D})}} \frac{\boldsymbol{\ell}!|c_{i,\boldsymbol{\ell}}|^2}{2d_i - 1},$$

$$\sup_{s\in[0,1]} \mathrm{E}\,\|Z_D(s) - Z_D^{(N)}(s)\|_{\mathbb{H}_2}^2 \leq C \sum_{\substack{i\geq 1,\ \boldsymbol{\ell}\in\mathcal{D}_i\\ (i,\boldsymbol{\ell})\notin\mathcal{F}_N(\mathcal{D})}} \frac{\boldsymbol{\ell}!|c_{i,\boldsymbol{\ell}}|^2}{2d_i - 1}.$$

For the second inequality, use the labelled isometry and $H(\boldsymbol{\ell}) = d_i$ on $\mathcal{D}_i$. By Parseval's identity and Assumption 3.2(ii), the full weighted coefficient sum is finite. Since the sets $\mathcal{F}_N(\mathcal{D})$ increase to all dominant indices, dominated convergence shows that both right-hand sides tend to zero as

$N \to \infty$. Summing over the m times and applying Markov's inequality verifies the approximation condition of [5, Theorem 3.2]. That theorem therefore yields

$$(A_n(s_a))_{a=1}^m \xrightarrow{d} (Z_D(s_a))_{a=1}^m \quad \text{in } \mathbb{H}_2^m. \tag{5.22}$$

We turn to the remainder $R_n(s)$ in (5.20). Proposition 5.2 implies, for every $\varepsilon > 0$,

$$\mathbb{P}\big(\|(R_n(s_a))_{a=1}^m\|_{\mathbb{H}_2^m} > \varepsilon\big) \le \varepsilon^{-2} \sum_{a=1}^m \mathrm{E}\,\|R_n(s_a)\|_{\mathbb{H}_2}^2 \le \varepsilon^{-2} m \sup_{s\in[0,1]} \mathrm{E}\,\|R_n(s)\|_{\mathbb{H}_2}^2 \to 0, \tag{5.23}$$

where the first inequality is Markov's inequality. Combining (5.20), (5.22), and (5.23), Slutsky's lemma in $\mathbb{H}_2^m$ proves (3.9).

Continuous interpolations: Recall (3.8), with $T_n = n^{-D}$. Since Assumption 3.2(i) gives $d_i > 1/2$ for every i, including the prescribed values on zero output directions, $\|n^{-D}\|_{\mathcal{L}(\mathbb{H}_2)} \le n^{-1/2}$. Stationarity therefore gives, for every fixed $s \in [0,1]$,

$$\mathrm{E}\left\|\widetilde{S}_n^T(s) - n^{-D}\sum_{t=1}^{\lfloor ns\rfloor} G[X_t]\right\|_{\mathbb{H}_2}^2 \le \|n^{-D}\|_{\mathcal{L}(\mathbb{H}_2)}^2\,\mathrm{E}\,\|G[X_0]\|_{\mathbb{H}_2}^2 \le n^{-1}\,\mathrm{E}\,\|G[X_0]\|_{\mathbb{H}_2}^2 \to 0.$$

Summing this estimate over $s_1, \dots, s_m$ shows that the interpolations have the same limiting finite-dimensional distributions.

Tightness: We apply the tightness criterion in [24, Proposition 4.1] with parameter space $[0,1]$ and state space $\mathbb{H}_2$. The marginal laws are tight at every fixed time by the convergence already proved. It remains to establish a uniform increment bound with a power strictly greater than one.

We impose the additional condition $d_* > 1/2$ in Theorem 3.1. For every integer a and every integer $m \ge 1$, stationarity and the finite-projection argument in the proof of Lemma 6.3 give

$$\begin{aligned}\mathrm{E}\left|\sum_{t=a+1}^{a+m} \langle G[X_t], v_i\rangle_{\mathbb{H}_2}\right|^2 &= \sum_{p=1}^{\infty}\sum_{\boldsymbol{\ell}\in\mathcal{L}_p} |c_{i,\boldsymbol{\ell}}|^2\,\mathrm{Var}\left(\sum_{t=1}^m H_{\boldsymbol{\ell}}[X_t]\right)\\ &\le \frac{C}{2d_i-1}\,\mathrm{E}\,|\langle G[X_0], v_i\rangle_{\mathbb{H}_2}|^2\, m^{2d_i}.\end{aligned} \tag{5.24}$$

For the inequality, every index with $c_{i,\boldsymbol{\ell}} \neq 0$ satisfies $H(\boldsymbol{\ell}) \le d_i$ by the definition of d_i and Assumption 3.2(i). We therefore apply (6.4) of Lemma 6.2, and then Parseval's identity for the scalar Hermite expansion of $\langle G, v_i\rangle$.

Let $U_{n,i}(s) = \langle \widetilde{S}_n^T(s), v_i\rangle_{\mathbb{H}_2}$, with $T_n = n^{-D}$, and put $h = t - s > 0$. We may take the constant C in (5.24) to be at least one. For grid times $s, t \in \{0, 1/n, \dots, 1\}$ with $s < t$, we have

$$\begin{aligned}\|U_{n,i}(t) - U_{n,i}(s)\|_{L^2(\Omega)} &= n^{-d_i}\left\|\sum_{r=ns+1}^{nt}\langle G[X_r], v_i\rangle_{\mathbb{H}_2}\right\|_{L^2(\Omega)}\\ &\le \left(\frac{C}{2d_i-1}\right)^{1/2}\|\langle G[X_0], v_i\rangle_{\mathbb{H}_2}\|_{L^2(\Omega)}\,h^{d_i},\end{aligned}$$

where the equality follows from the definition of the interpolation at grid times. The inequality is (5.24) with block length $n(t-s)$. For arbitrary s, t with $h \ge 1/n$, put $j = \lfloor ns\rfloor$ and $k = \lfloor nt\rfloor$,

so that $k \geq j+1$. We split the increment as follows

$$\begin{aligned}
&\|U_{n,i}(t) - U_{n,i}(s)\|_{L^2(\Omega)} \\
&\leq \|U_{n,i}(t) - U_{n,i}(k/n)\|_{L^2(\Omega)} + \|U_{n,i}(k/n) - U_{n,i}((j+1)/n)\|_{L^2(\Omega)} \\
&\qquad\qquad + \|U_{n,i}((j+1)/n) - U_{n,i}(s)\|_{L^2(\Omega)} \\
&\leq \left(\frac{C}{2d_i - 1}\right)^{1/2} \|\langle G[X_0], v_i\rangle_{\mathbb{H}_2}\|_{L^2(\Omega)} \left[\frac{nt-k}{n^{d_i}} + \left(\frac{k-j-1}{n}\right)^{d_i} + \frac{j+1-ns}{n^{d_i}}\right] \\
&\leq 3\left(\frac{C}{2d_i - 1}\right)^{1/2} \|\langle G[X_0], v_i\rangle_{\mathbb{H}_2}\|_{L^2(\Omega)}\, h^{d_i}.
\end{aligned}$$

The first inequality is the triangle inequality. For the second, linear interpolation and the case $m = 1$ of (5.24) control the two partial increments, while the grid-time estimate controls the middle block. For the last inequality, each partial-increment coefficient is at most one, $n^{-d_i} \leq h^{d_i}$, and $(k-j-1)/n \leq h$.

If $0 < h < 1/n$, we use the piecewise constant slope of the linear interpolation to get, with further explanations below,

$$\begin{aligned}
\|U_{n,i}(t) - U_{n,i}(s)\|_{L^2(\Omega)} &= n^{1-d_i} \left\| \int_s^t \langle G[X_{\lfloor nv\rfloor+1}], v_i\rangle_{\mathbb{H}_2}\, dv \right\|_{L^2(\Omega)} \\
&\leq n^{1-d_i} \int_s^t \|\langle G[X_{\lfloor nv\rfloor+1}], v_i\rangle_{\mathbb{H}_2}\|_{L^2(\Omega)}\, dv \\
&= n^{1-d_i} h\, \|\langle G[X_0], v_i\rangle_{\mathbb{H}_2}\|_{L^2(\Omega)} \\
&\leq h^{d_i} \|\langle G[X_0], v_i\rangle_{\mathbb{H}_2}\|_{L^2(\Omega)}.
\end{aligned}$$

The first equality follows from the interpolation formula, the first inequality is Minkowski's integral inequality, and the next equality uses stationarity. The following inequality factors out h^{d_i}, and uses $nh < 1$ and $d_i < 1$, the latter by Assumption 3.2(i). Since $C \geq 1$ and $2d_i - 1 < 1$, both cases are bounded by the estimate with factor $3(C/(2d_i-1))^{1/2}$ above. Consequently,

$$\begin{aligned}
\mathrm{E}\,\|\widetilde{S}_n^T(t) - \widetilde{S}_n^T(s)\|_{\mathbb{H}_2}^2 &= \sum_{i=1}^{\infty} \|U_{n,i}(t) - U_{n,i}(s)\|_{L^2(\Omega)}^2 \\
&\leq 9C \sum_{i=1}^{\infty} \frac{\mathrm{E}\,|\langle G[X_0], v_i\rangle_{\mathbb{H}_2}|^2}{2d_i - 1} |t-s|^{2d_i} \qquad (5.25) \\
&\leq 9C|t-s|^{2d_*} \sum_{i=1}^{\infty} \frac{\mathrm{E}\,|\langle G[X_0], v_i\rangle_{\mathbb{H}_2}|^2}{2d_i - 1}. \qquad (5.26)
\end{aligned}$$

The equality follows from Parseval's identity and Tonelli's theorem. The inequality (5.25) uses the preceding two cases, and (5.26) uses $|t-s| \leq 1$ and $d_i \geq d_*$ on active coordinates. Inactive coordinates contribute zero. The coefficient sum is finite by Assumption 3.2(ii).

Since $2d_* > 1$, [24, Proposition 4.1] now gives tightness in $C([0,1]; \mathbb{H}_2)$. Its covering-number condition holds on $[0,1]$ with exponent one; the increment moment has order two and time exponent $2d_*$. The state space is complete and separable, and the interpolations have continuous paths, so all its hypotheses are satisfied.

For completeness, the same increment estimate also gives a continuous version of the limiting process. By the finite-dimensional convergence and the Portmanteau theorem,

$$\mathrm{E}\,\|Z_D(t) - Z_D(s)\|_{\mathbb{H}_2}^2 \leq 9C|t-s|^{2d_*} \sum_{i=1}^{\infty} \frac{\mathrm{E}\,|\langle G[X_0], v_i\rangle_{\mathbb{H}_2}|^2}{2d_i - 1}.$$

Hence [24, Theorem 1.1] supplies a continuous modification. Finally, tightness and the finite-dimensional convergence, together with [23, Lemma 16.2], yield

$$\widetilde{S}_n^T \xrightarrow{d} Z_D \quad \text{in } C([0,1];\mathbb{H}_2),$$

where Z_D denotes this continuous version. □

5.4 Proof of Corollary 3.1

Proof. Recall the expansion $G[x] = \sum_{i\geq 1}\sum_{\ell\in\mathcal{L}} c_{i,\ell}H_\ell[x]\otimes v_i$ from (2.3). We aim to apply Theorem 3.1 after expanding $G[x]$ with respect to a basis that is possibly different than $(v_i)_i$ and in which every coordinate of G has a nonzero term of order q. We then verify the assumption do Theorem 3.1. For every $\boldsymbol{\ell}\in\mathcal{L}$, denote by

$$c_{\boldsymbol{\ell}} \doteq \sum_{j=1}^{\infty} c_{j,\boldsymbol{\ell}} v_j \in \mathbb{H}_2$$

the coefficient vector in (2.3). Since G has Hermite rank q, there exists $\boldsymbol{\ell}_0\in\mathcal{L}_q$ such that $c_{\boldsymbol{\ell}_0}\neq 0$. To construct the required basis, choose $(a_i)_{i\geq 1}\in\ell^2$ with $a_i>0$ for every $i\geq 1$ and $\sum_i a_i^2 = 1$, and put $u=\sum_i a_i v_i$. By extending u and $c_{\boldsymbol{\ell}_0}/\|c_{\boldsymbol{\ell}_0}\|_{\mathbb{H}_2}$ to orthonormal bases, we obtain a surjective linear isometry U of $\mathbb{H}_2$ satisfying $Uu = c_{\boldsymbol{\ell}_0}/\|c_{\boldsymbol{\ell}_0}\|_{\mathbb{H}_2}$. Let $w_i = Uv_i$, and denote by $\widetilde{c}_{i,\boldsymbol{\ell}}$ the Hermite coefficients in the orthonormal basis $(w_i)_{i\geq 1}$. For every $i\geq 1$ and $\boldsymbol{\ell}\in\mathcal{L}$, we have

$$\begin{aligned} \widetilde{c}_{i,\boldsymbol{\ell}} = \langle c_{\boldsymbol{\ell}}, w_i\rangle_{\mathbb{H}_2} = \sum_{j=1}^{\infty} c_{j,\boldsymbol{\ell}}\langle v_j, w_i\rangle_{\mathbb{H}_2}, \quad \widetilde{c}_{i,\boldsymbol{\ell}_0} = \|c_{\boldsymbol{\ell}_0}\|_{\mathbb{H}_2}\langle Uu, Uv_i\rangle_{\mathbb{H}_2} = \|c_{\boldsymbol{\ell}_0}\|_{\mathbb{H}_2} a_i > 0, \\ \sum_{i=1}^{\infty} |\widetilde{c}_{i,\boldsymbol{\ell}}|^2 = \|c_{\boldsymbol{\ell}}\|_{\mathbb{H}_2}^2 = \sum_{j=1}^{\infty} |c_{j,\boldsymbol{\ell}}|^2, \end{aligned} \tag{5.27}$$

where the representation of $\widetilde{c}_{i,\boldsymbol{\ell}_0}$ uses the isometry of U, and the last line follows from Parseval's identity. In this basis,

$$G[X_t] = \sum_{i=1}^{\infty}\langle G[X_t], w_i\rangle_{\mathbb{H}_2} w_i = \sum_{i=1}^{\infty}\left(\sum_{\boldsymbol{\ell}\in\mathcal{L}}\widetilde{c}_{i,\boldsymbol{\ell}}H_{\boldsymbol{\ell}}[X_t]\right) w_i = \sum_{i=1}^{\infty}\sum_{\boldsymbol{\ell}\in\mathcal{L}}\widetilde{c}_{i,\boldsymbol{\ell}}H_{\boldsymbol{\ell}}[X_t]\, w_i,$$

with convergence in $L^2(\Omega;\mathbb{H}_2)$. Thus the output basis and coefficients are replaced by w_i and $\widetilde{c}_{i,\boldsymbol{\ell}}$, respectively. Moreover, the coefficient norm identity above shows that a multi-index is active in the new basis if and only if it is active in the original basis. Consequently, the active input coordinates are unchanged, and Assumption 3.1 remains valid.

To verify Assumption 3.2 in the basis (w_i), denote by $\widetilde{d}_i$, $\widetilde{\mathcal{D}}_i$, and $\widetilde{D}$ the corresponding dominant exponents, dominant sets, and operator, respectively, as defined in (3.4) and (3.7). By Assumption 3.3(ii), all input coordinates in the support of an active index of order q have exponent α. Hence, for every $\boldsymbol{\ell}\in\mathcal{L}_q$ with $c_{\boldsymbol{\ell}}\neq 0$,

$$\beta(\boldsymbol{\ell}) = \sum_j \ell_j(1-\alpha_j) = (1-\alpha)|\boldsymbol{\ell}| = q(1-\alpha), \quad H(\boldsymbol{\ell}) = 1-\frac{q}{2}(1-\alpha) = H\in(1/2,1),$$

and $K_{s,\boldsymbol{\ell}} = K_s$ by (4.2). Moreover, Assumption 3.3(iii) gives $H(\boldsymbol{\ell})<H$ for every active higher-order index. Since $\widetilde{c}_{i,\boldsymbol{\ell}_0}>0$ for every $i\geq 1$, the dominant exponent in each output coordinate is attained at $\boldsymbol{\ell}_0$. Consequently,

$$\widetilde{d}_i = H, \qquad \widetilde{\mathcal{D}}_i = \{\boldsymbol{\ell}\in\mathcal{L}_q : \widetilde{c}_{i,\boldsymbol{\ell}}\neq 0\}, \qquad \widetilde{D} = HI_{\mathbb{H}_2},$$

which verifies Assumption 3.2(i). Furthermore, Parseval's identity and Assumption 3.3(i) yield

$$\sum_{i=1}^{\infty} \frac{\mathrm{E}\,|\langle G[X_0], w_i\rangle_{\mathbb{H}_2}|^2}{2\widetilde{d}_i - 1} = \frac{1}{2H-1}\sum_{i=1}^{\infty}\sum_{\boldsymbol{\ell}\in\mathcal{L}} \boldsymbol{\ell}!|\widetilde{c}_{i,\boldsymbol{\ell}}|^2$$
$$= \frac{1}{2H-1}\sum_{\boldsymbol{\ell}\in\mathcal{L}} \boldsymbol{\ell}!\|c_{\boldsymbol{\ell}}\|_{\mathbb{H}_2}^2 = \frac{\mathrm{E}\,\|G[X_0]\|_{\mathbb{H}_2}^2}{2H-1} < \infty,$$

which verifies Assumption 3.2(ii).

Since $\widetilde{D} = HI_{\mathbb{H}_2}$ and $\inf_i \widetilde{d}_i = H > 1/2$, Theorem 3.1, applied in the basis $(w_i)_i$, gives both finite-dimensional convergence and weak convergence of the continuous interpolations with $T_n = n^{-H} I_{\mathbb{H}_2}$. Denote by $\widetilde{Z}$ the resulting limit. It remains to identify $\widetilde{Z}$ with $Z_H^{(q)}$. For every $s \in [0,1]$, (4.8) and the kernel identity above give

$$\widetilde{Z}(s) = \sum_{i=1}^{\infty}\sum_{\boldsymbol{\ell}\in\mathcal{L}_q} \widetilde{c}_{i,\boldsymbol{\ell}}\kappa^{\boldsymbol{\ell}/2} I_q(J_{\boldsymbol{\ell}}K_s)w_i = \sum_{\boldsymbol{\ell}\in\mathcal{L}_q} \kappa^{\boldsymbol{\ell}/2} I_q(J_{\boldsymbol{\ell}}K_s)\left(\sum_{i=1}^{\infty}\widetilde{c}_{i,\boldsymbol{\ell}}w_i\right)$$
$$= \sum_{\boldsymbol{\ell}\in\mathcal{L}_q} \kappa^{\boldsymbol{\ell}/2} I_q(J_{\boldsymbol{\ell}}K_s)\left(\sum_{i=1}^{\infty}\langle c_{\boldsymbol{\ell}}, w_i\rangle_{\mathbb{H}_2} w_i\right) = \sum_{\boldsymbol{\ell}\in\mathcal{L}_q} \kappa^{\boldsymbol{\ell}/2} I_q(J_{\boldsymbol{\ell}}K_s)c_{\boldsymbol{\ell}}$$
$$= \sum_{\boldsymbol{\ell}\in\mathcal{L}_q} \kappa^{\boldsymbol{\ell}/2} I_q(J_{\boldsymbol{\ell}}K_s)\left(\sum_{j=1}^{\infty} c_{j,\boldsymbol{\ell}}v_j\right) = \sum_{j=1}^{\infty}\sum_{\boldsymbol{\ell}\in\mathcal{L}_q} c_{j,\boldsymbol{\ell}}\kappa^{\boldsymbol{\ell}/2} I_q(J_{\boldsymbol{\ell}}K_s)v_j = Z_H^{(q)}(s),$$

where we used (5.27). This identification yields (3.10) and weak convergence in $C([0,1];\mathbb{H}_2)$ to a continuous version of $Z_H^{(q)}$. □

6 Additional results and their proofs

The following lemma provides a spectral comparison for Hermite partial sums.

Lemma 6.1. *Suppose Assumptions 3.1(ii)–(iv). There exists a constant $C \geq 1$, depending only on M, κ_* and $\sup_j \alpha_j$, such that, for every $\boldsymbol{\ell} \in \mathcal{L}_p$, $p \geq 1$, supported on active coordinates, and all integers $m, r \geq 1$,*

$$\mathrm{Var}\left(\sum_{t=1}^{m} H_{\boldsymbol{\ell}}[X_t]\right) \leq \boldsymbol{\ell}! C^{p\wedge r} \sum_{|h|<m} (m-|h|)(1+|h|)^{-\sum_{a=1}^{p\wedge r}(1-\alpha_{k_a})}, \tag{6.1}$$

where $\mathbf{k} = \mathbf{k}(\boldsymbol{\ell})$.

Proof. By Assumption 3.1(iii), we have $\widetilde{f}_j(\omega) \leq g_j(\omega) \doteq M\kappa_j|\omega|^{-\alpha_j}$ for every active coordinate j. For $u \in L^1([-\pi,\pi])$, write

$$\widehat{u}(h) = \int_{-\pi}^{\pi} e^{ih\omega} u(\omega)\, d\omega.$$

We first note that, uniformly over active coordinates,

$$|\widehat{g}_j(h)| \leq C(1+|h|)^{-(1-\alpha_j)}, \qquad |\widehat{\widetilde{f}}_j(h)| \leq 1, \qquad h \in \mathbb{Z}. \tag{6.2}$$

Indeed, direct integration gives $\widehat{g}_j(0) = 2M\kappa_j\pi^{1-\alpha_j}/(1-\alpha_j)$. For $h \neq 0$, evenness and a change of variables give

$$|\widehat{g}_j(h)| = 2M\kappa_j|h|^{\alpha_j-1}\left|\int_0^{\pi|h|} u^{-\alpha_j}\cos u\, du\right| \leq 2M\kappa_j|h|^{\alpha_j-1}\left(\frac{1}{1-\alpha_j}+2\right).$$

The last inequality follows by splitting the integral at 1 and integrating by parts on $[1, \pi|h|]$. The bounds $\kappa_j \le \kappa_*$ from Assumption 3.1(iv) and $\sup_j \alpha_j < 1$ from Assumption 3.1(iii) yield the first estimate in (6.2). The second follows from $\int_{-\pi}^{\pi} \widetilde{f}_j(\omega)\, d\omega = 1$.

Put $q = p \wedge r$ and define $D_m(y) = \sum_{t=1}^{m} e^{ity}$. Expanding the variance and substituting the Fourier representations yields

$$\begin{aligned}
\operatorname{Var}\left(\sum_{t=1}^{m} H_{\boldsymbol{\ell}}[X_t]\right) &= \boldsymbol{\ell}! \int_{[-\pi,\pi]^p} |D_m(\omega_1 + \cdots + \omega_p)|^2 \prod_{a=1}^{p} \widetilde{f}_{k_a}(\omega_a)\, d\boldsymbol{\omega} \\
&\le \boldsymbol{\ell}! \int_{[-\pi,\pi]^p} |D_m(\omega_1 + \cdots + \omega_p)|^2 \prod_{a=1}^{q} g_{k_a}(\omega_a) \prod_{a=q+1}^{p} \widetilde{f}_{k_a}(\omega_a)\, d\boldsymbol{\omega} \\
&\le \boldsymbol{\ell}! \sum_{|h|<m} (m - |h|) \prod_{a=1}^{q} |\widehat{g}_{k_a}(h)| \prod_{a=q+1}^{p} |\widehat{\widetilde{f}}_{k_a}(h)|.
\end{aligned}$$

The first inequality follows from the nonnegativity of the integrand. For the second, we expand $|D_m|^2$, integrate each frequency separately, and take absolute values; there are $m - |h|$ pairs of time indices with difference h. All interchanges involve finite sums and integrable functions, and empty products are understood to equal one. Applying (6.2) proves (6.1). □

The following lemma employs the bounds of the previous lemma to express the variance of the Hermite partial sums in terms of $\beta = \beta(\boldsymbol{\ell})$ from (3.3).

Lemma 6.2. *Suppose Assumptions 3.1(ii)–(iv). There exists a constant $C \ge 1$, depending only on M, κ_* and $\sup_j \alpha_j$, such that, for every $\boldsymbol{\ell} \in \mathcal{L}_p$, $p \ge 1$, supported on active coordinates, and every integer $m \ge 1$,*

$$\operatorname{Var}\left(\sum_{t=1}^{m} H_{\boldsymbol{\ell}}[X_t]\right) \le \boldsymbol{\ell}! C^p \begin{cases} (1 + 2/(1-\beta))m^{2-\beta}, & 0 < \beta < 1, \\ m(1 + 2\log m), & \beta = 1, \\ (1 + 2/(\beta - 1))m, & \beta > 1, \end{cases} \tag{6.3}$$

where $\beta = \beta(\boldsymbol{\ell})$. Moreover, for every $d \in (1/2, 1)$ satisfying $H(\boldsymbol{\ell}) \le d$,

$$\operatorname{Var}\left(\sum_{t=1}^{m} H_{\boldsymbol{\ell}}[X_t]\right) \le \frac{C\boldsymbol{\ell}!}{2d-1}\, m^{2d}. \tag{6.4}$$

Proof. Under Assumptions 3.1(ii)–(iv), let $C_0 \ge 1$ be the constant in Lemma 6.1. Fix $\boldsymbol{\ell} \in \mathcal{L}_p$ supported on active coordinates, and write $\mathbf{k} = \mathbf{k}(\boldsymbol{\ell})$ and $\beta = \beta(\boldsymbol{\ell})$. Applying that lemma with $r = p$ gives the lag-sum estimate used in [12, proof of Lemma 3.1]:

$$\begin{aligned}
\operatorname{Var}\left(\sum_{t=1}^{m} H_{\boldsymbol{\ell}}[X_t]\right) &\le \boldsymbol{\ell}! C_0^p m \left(1 + 2\sum_{h=1}^{m-1} (1+h)^{-\beta}\right) \\
&\le \boldsymbol{\ell}! C_0^p m \left(1 + 2\int_1^m x^{-\beta}\, dx\right).
\end{aligned} \tag{6.5}$$

The first inequality separates the zero lag and uses $m - h \le m$. The second follows because $x^{-\beta}$ is decreasing. Evaluating the integral gives the three cases in (6.3), with C_0 in place of C; for $0 < \beta < 1$, we also use $m \le m^{2-\beta}$.

For uniformity in the chaos order, we use fixed-order domination as in [3, equation (37)]. Put $\varepsilon = 1 - \sup_j \alpha_j > 0$ by Assumption 3.1(iii), and choose an integer r_0 depending only on ε

such that $r_0\varepsilon > 1$. For $p \geq r_0$, Lemma 6.1 with $r = r_0$ gives

$$\operatorname{Var}\left(\sum_{t=1}^{m} H_{\boldsymbol{\ell}}[X_t]\right) \leq \boldsymbol{\ell}! C_0^{r_0} m \left(1 + 2\sum_{h=1}^{\infty}(1+h)^{-r_0\varepsilon}\right)$$
$$\leq \boldsymbol{\ell}! C_0^{r_0} m \left(1 + \frac{2}{r_0\varepsilon - 1}\right) \doteq C_1 \boldsymbol{\ell}! m. \tag{6.6}$$

Here we used $\sum_{a=1}^{r_0}(1-\alpha_{k_a}) \geq r_0\varepsilon$ and the integral comparison on $[1,\infty)$. The constant C_1 depends only on M, κ_* and $\sup_j \alpha_j$.

Finally, let $d \in (1/2, 1)$ satisfy $H(\boldsymbol{\ell}) \leq d$. For $p < r_0$, we have $\beta = 2 - 2H(\boldsymbol{\ell}) \geq 2 - 2d$, so the first bound in (6.5) yields

$$\operatorname{Var}\left(\sum_{t=1}^{m} H_{\boldsymbol{\ell}}[X_t]\right) \leq \boldsymbol{\ell}! C_0^{r_0} m \left(1 + 2\int_1^m x^{2d-2}\,dx\right)$$
$$= \boldsymbol{\ell}! C_0^{r_0} m \left(1 + 2\frac{m^{2d-1}-1}{2d-1}\right) \leq \frac{3C_0^{r_0}\boldsymbol{\ell}!}{2d-1} m^{2d}.$$

We used $C_0^p \leq C_0^{r_0}$ and $0 < 2d - 1 < 1$. For $p \geq r_0$, (6.6) gives

$$\operatorname{Var}\left(\sum_{t=1}^{m} H_{\boldsymbol{\ell}}[X_t]\right) \leq C_1 \boldsymbol{\ell}! m \leq \frac{C_1 \boldsymbol{\ell}!}{2d-1} m^{2d}.$$

Choosing $C = \max\{C_0, C_1, 3C_0^{r_0}\}$ proves both (6.3) and (6.4). □

The following lemma provides a uniform bound for the projection remainder.

Lemma 6.3. *Suppose Assumptions 3.1(ii)–(iv) and $H(\boldsymbol{\ell}) \leq d_i$ for every $i \geq 1$ and $\boldsymbol{\ell} \in \mathcal{Q}_i$. Then there is a constant $C > 0$, such that, for all $n, N \geq 1$,*

$$\sup_{s\in[0,1]} \mathrm{E}\left\| n^{-D} \sum_{t=1}^{\lfloor ns \rfloor} (G_{\mathcal{Q}} - G_{\mathcal{Q}}^{(N)})[X_t] \right\|_{\mathbb{H}_2}^2 \leq C \sum_{\substack{i\geq 1,\ \boldsymbol{\ell}\in\mathcal{Q}_i \\ (i,\boldsymbol{\ell})\notin\mathcal{F}_N(\mathcal{Q})}} \frac{\boldsymbol{\ell}! |c_{i,\boldsymbol{\ell}}|^2}{2d_i - 1}. \tag{6.7}$$

Proof. For $L \geq N$ and $0 \leq m \leq n$, Lemma 2.1, gives

$$\mathrm{E}\left\| n^{-D} \sum_{t=1}^{m} (G_{\mathcal{Q}}^{(L)} - G_{\mathcal{Q}}^{(N)})[X_t] \right\|_{\mathbb{H}_2}^2$$
$$= \sum_{(i,\boldsymbol{\ell})\in\mathcal{F}_L(\mathcal{Q})\setminus\mathcal{F}_N(\mathcal{Q})} |c_{i,\boldsymbol{\ell}}|^2 n^{-2d_i} \operatorname{Var}\left(\sum_{t=1}^{m} H_{\boldsymbol{\ell}}[X_t]\right) \leq C \sum_{\substack{i\geq 1,\ \boldsymbol{\ell}\in\mathcal{Q}_i \\ (i,\boldsymbol{\ell})\notin\mathcal{F}_N(\mathcal{Q})}} \frac{\boldsymbol{\ell}! |c_{i,\boldsymbol{\ell}}|^2}{2d_i - 1}.$$

For the inequality, apply (6.4) under Assumptions 3.1(ii)–(iv) with $d = d_i$ and use $(m/n)^{2d_i} \leq 1$; the case $m = 0$ is immediate. For fixed n and m, stationarity, boundedness of n^{-D} and the triangle inequality imply that the corresponding partial sums converge in $L^2(\Omega; \mathbb{H}_2)$ as $L \to \infty$, since $G_{\mathcal{Q}}^{(L)} \to G_{\mathcal{Q}}$ in $L^2(\mathbb{H}_1, \gamma_{\mathcal{Q}}; \mathbb{H}_2)$. Passing to the limit in the displayed estimate and taking the supremum over $m = \lfloor ns \rfloor$, $s \in [0,1]$, proves (6.7). □

7 Applications

7.1 Nonlinear transformations of a functional FARIMA process

Fractionally integrated autoregressive moving average (FARIMA) models are a popular class that exhibit long-range dependence. The scalar construction goes back to [20, 21]. Functional FARIMA models were introduced in [31]. In this section, we apply Corollary 3.1 to nonlinear transformations of a functional FARIMA process. These nonlinear transformations may include Sample moment operators, sample eigenvalues or neural operators; see, for instance Section 7 in [17] for examples.

Let Φ and Θ be polynomials with real coefficients, $\Phi(0) = \Theta(0) = 1$, and no zeros in $\{z : |z| \le 1\}$. Fix $d \in (0, 1/2)$ and set $\alpha = 2d$. With respect to the basis $\{\phi_k\}_{k\ge1}$ of $\mathbb{H}_1$ used above, let $\{\varepsilon_t : t \in \mathbb{Z}\}$ be independent centered Gaussian random variables with covariance operator $\Sigma_\varepsilon \phi_k = \nu_k \phi_k$, where

$$\nu_k > 0 \quad \text{for every } k \ge 1, \qquad \sum_{k\ge1} \nu_k < \infty.$$

We consider the stationary causal solution of

$$\Phi(B)(1-B)^d X_t = \Theta(B)\varepsilon_t, \qquad t \in \mathbb{Z}, \tag{7.1}$$

where B denotes the backward-shift operator and the fractional difference is defined by its binomial series.

To identify the spectral quantities in Assumption 3.1, define

$$\begin{aligned} g_d(\omega) &= \frac{1}{2\pi}\left|\frac{\Theta(e^{-i\omega})}{\Phi(e^{-i\omega})}\right|^2 |1-e^{-i\omega}|^{-\alpha}, \qquad 0 < |\omega| \le \pi, \\ v_d &= \int_{-\pi}^{\pi} g_d(\omega)\, d\omega, \qquad\qquad \kappa = \frac{1}{2\pi v_d}\left|\frac{\Theta(1)}{\Phi(1)}\right|^2. \end{aligned} \tag{7.2}$$

We verify below that $0 < v_d < \infty$. The covariance eigenvalues are $\lambda_k = v_d \nu_k$, and the standardized coordinate processes $\{W_{\phi_k}(X_t) : t \in \mathbb{Z}\}$, defined in (2.1), are independent copies of a scalar Gaussian FARIMA process with variance one. For a function $G \in L^2(\mathbb{H}_1, \gamma_Q; \mathbb{H}_2)$, recall its Hermite coefficients, rank q, and chaos projections G_p from (2.3)–(2.4).

Proposition 7.1. *Let $\{X_t : t \in \mathbb{Z}\}$ satisfy (7.1), and let $G \in L^2(\mathbb{H}_1, \gamma_Q; \mathbb{H}_2)$ be centered and nonzero. Suppose that $q(1-\alpha) < 1$, and let H be as in (3.6). Then the interpolated partial-sum process $\widetilde{S}_n$ in (3.8), with $T_n = n^{-H} I_{\mathbb{H}_2}$, satisfies*

$$\widetilde{S}_n \xrightarrow{d} Z_H^{(q)} \quad \text{in } C([0,1]; \mathbb{H}_2), \qquad n \to \infty,$$

where $Z_H^{(q)}$ is given in (4.10) with $\kappa_k = \kappa$ for every $k \ge 1$. The process S_n in (3.8) converges to the same limit in finite-dimensional distributions in $\mathbb{H}_2$.

Proof. We obtain the convergence by means of Corollary 3.1. We need to verify Assumptions 3.1 and 3.3. The root conditions on Φ and Θ imply that $\left|\Theta(e^{-i\omega})/\Phi(e^{-i\omega})\right|^2$ is bounded above and away from zero on $[-\pi, \pi]$. Moreover,

$$\frac{2}{\pi}|\omega| \le |1-e^{-i\omega}| \le |\omega|, \qquad |\omega| \le \pi.$$

Consequently, v_d in (7.2) satisfies $0 < v_d < \infty$, since $2d = \alpha \in (0,1)$.

Since Φ has no zeros in the closed unit disc, the rational function Θ/Φ is analytic on a neighborhood of that disc and bounded there. The binomial series defining $(1-z)^{-d}$ converges for $|z|<1$. Hence the transfer function

$$A(z) := \frac{\Theta(z)}{\Phi(z)}(1-z)^{-d} = \sum_{j=0}^{\infty} a_j z^j, \qquad |z|<1,$$

is analytic on the open unit disc, with real coefficients a_j.

Since $0<d<1/2$ and Φ has no zeros in the closed unit disc, the scalar FARIMA representation implies that $(a_j)_{j\geq 0}$ is square summable and that

$$\sum_{j=0}^{\infty} a_j e^{-ij\omega} = \frac{\Theta(e^{-i\omega})}{\Phi(e^{-i\omega})}(1-e^{-i\omega})^{-d} \quad \text{in } L^2(-\pi,\pi);$$

see Section 2.4.2 in [42]. Consequently, Parseval's identity gives

$$\sum_{j=0}^{\infty} a_j^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\Theta(e^{-i\omega})}{\Phi(e^{-i\omega})}\right|^2 |1-e^{-i\omega}|^{-2d}\, d\omega = v_d < \infty.$$

Since the innovations are independent and centered, the series $X_t = \sum_{j\geq 0} a_j \varepsilon_{t-j}$ converges in $L^2(\Omega;\mathbb{H}_1)$, because

$$\sum_{j\geq 0} \mathrm{E}\,\|a_j\varepsilon_{t-j}\|_{\mathbb{H}_1}^2 = v_d \sum_{k\geq 1}\nu_k < \infty.$$

It defines a centered stationary Gaussian solution of (7.1), with

$$\mathscr{F}_\omega = g_d(\omega)\Sigma_\varepsilon, \qquad Q = v_d\Sigma_\varepsilon, \qquad f_k(\omega) = \nu_k g_d(\omega). \tag{7.3}$$

In particular, $\lambda_k = v_d\nu_k$ for every $k\geq 1$. Since Σ_ε is trace-class, $\mathscr{F}_\omega$ is trace-class for almost every $\omega\in[-\pi,\pi]$, and

$$\int_{-\pi}^{\pi} \mathrm{Tr}(\mathscr{F}_\omega)\, d\omega = v_d\sum_{k\geq 1}\nu_k < \infty.$$

This verifies Assumption 3.1(i). Furthermore, the spectral density and autocovariance operators are diagonal in the basis $\{\phi_k\}_{k\geq 1}$. Their spectral densities f_k are nonnegative and even, where evenness follows from the real coefficients of Φ and Θ. Thus Assumption 3.1(ii) also holds.

For Assumption 3.1(iii), take $\alpha_k=\alpha$, $\kappa_k=\kappa$, and

$$L_k(r) = \left|\frac{\Theta(e^{-ir})\Phi(1)}{\Phi(e^{-ir})\Theta(1)}\right|^2 \left(\frac{r}{2\sin(r/2)}\right)^{\alpha}, \qquad 0<r\leq\pi, \quad k\geq 1.$$

Then (3.2) follows from (7.2) and (7.3). The functions L_k are nonnegative, identical for all k, and extend continuously to $[0,\pi]$, with value one at zero. Consequently, $L_k(r)\to 1$ as $r\downarrow 0$, and there exists a finite constant M such that $L_k(r)\leq M$ for every $k\geq 1$ and $0<r\leq\pi$. This verifies Assumption 3.1(iii).

To verify Assumption 3.1(iv), note that

$$\frac{f_k(\omega)}{\lambda_k} = \frac{\nu_k g_d(\omega)}{\nu_k v_d} = \frac{g_d(\omega)}{v_d}, \qquad k\geq 1.$$

Thus all coordinates have the same standardized spectral density and, in particular, the same spectral scale $\kappa_k=\kappa$. Consequently,

$$\kappa_* = \kappa = \frac{1}{2\pi v_d}\left|\frac{\Theta(1)}{\Phi(1)}\right|^2 < \infty,$$

where finiteness follows from $v_d > 0$ and $\Phi(1) \neq 0$. This completes the verification of Assumption 3.1.

We now turn to Assumption 3.3. Square integrability of G and its Hermite rank q are part of the hypotheses. Hence Assumption 3.3(i) is satisfied.

For Assumption 3.3(ii), recall that $\alpha_j = \alpha$ for every input coordinate $j \geq 1$. In particular, whenever $c_{i,\boldsymbol{\ell}} \neq 0$ with $\boldsymbol{\ell} \in \mathcal{L}_q$, we have $\alpha_j = \alpha$ for every $j \in \operatorname{supp} \boldsymbol{\ell}$. Moreover, the assumed condition $q(1-\alpha) < 1$ is equivalent to $\alpha > 1 - 1/q$. Thus Assumption 3.3(ii) holds.

Finally, by (3.3), for every $\boldsymbol{\ell} \in \mathcal{L}_p$ with $p > q$,

$$H(\boldsymbol{\ell}) = 1 - \frac{p}{2}(1-\alpha) < 1 - \frac{q}{2}(1-\alpha) = H,$$

where the strict inequality follows from $\alpha < 1$. This verifies Assumption 3.3(iii). The conclusion now follows from Corollary 3.1. □

For Hilbert-valued linear processes, [11, 14] establish functional central limit theorems with Gaussian limits and operator normalizations. Their results concern partial sums of the linear process itself and allow innovations that need not be Gaussian. In the present FARIMA setting, taking $G[x] = x$ gives $q = 1$ and $H = 1/2 + d$, consistent with the scalar normalization in this linear limit theory. For general nonlinear transformations, the normalization depends additionally on the Hermite rank, as specified in (3.6).

A related functional FARIMA construction is given in [31, Section 4], where scalar fractional integration is applied to a functional ARMA process. Their Theorem 1 and Corollary 1 establish Gaussian limits for temporal sums of curve time series through a decomposition into finite-dimensional long-memory subspaces. In particular, their FARIMA application imposes a finite-rank condition on the covariance governing the dominant component. In our application, infinitely many input coordinates may share the same memory exponent, and we consider the partial-sum process of a nonlinear transformation with values in $\mathbb{H}_2$.

The input process considered here also belongs to the operator-valued FIARMA framework of [18]. Indeed, we obtain (7.1) by taking their fractional integration operator to be $dI_{\mathbb{H}_1}$, their autoregressive and moving-average polynomials to be $\Phi(z)I_{\mathbb{H}_1}$ and $\Theta(z)I_{\mathbb{H}_1}$, respectively, and their innovations to be Gaussian. Their framework permits more general operator coefficients and provides existence results for normal memory operators, together with prediction and estimation theory. Our proposition establishes a nonlinear functional limit theorem for the Gaussian subclass specified above.

Hilbert-valued nonlinear noncentral limit theorems have also been obtained for scalar Gaussian inputs. In [4, Theorem 2.7 of version 3], a scalar long-memory Gaussian sequence is transformed into an observation with values in a weighted L^2 space. Their reduction principle identifies the leading Hermite term and yields convergence of the normalized sum at time one. The limit is a deterministic coefficient function multiplied by a scalar Hermite random variable and therefore lies in a single deterministic output direction. Thus infinite-dimensional output is already covered by that result. In the present setting, the input may also be infinite-dimensional, and mixed Hermite terms involving different input coordinates may survive in the limit. Consequently, the limiting covariance need not have finite rank. Moreover, under our FARIMA assumptions, we obtain convergence of the interpolated partial-sum process in $C([0,1];\mathbb{H}_2)$.

7.2 Excursion areas of spherical Gaussian fields

We return to the sojourn functionals that motivated this work and apply Theorem 3.1 to a spherical Gaussian field sampled at integer times. We first identify the dominant Hermite indices of the excursion area, then state the resulting noncentral limit theorem, and finally compare it with the continuous-time results of Marinucci, Rossi and Vidotto [34].

7.2.1 The spherical model and the excursion functional

We use the spherical random wave setting presented in [34]. Let $\{Y_{\ell,m} : \ell \geq 0,\ |m| \leq \ell\}$ be the standard real orthonormal basis of $\mathbb{H}_1 = L^2(\mathbb{S}^2, dx)$ consisting of spherical harmonics, where dx is surface measure with total mass 4π and $Y_{0,0} = (4\pi)^{-1/2}$. Let $\{Z(x,t) : x \in \mathbb{S}^2,\ t \in \mathbb{R}\}$ be a jointly measurable, centered Gaussian field, stationary in time and isotropic in space, admitting the representation

$$Z(x,t) = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} a_{\ell,m}(t) Y_{\ell,m}(x), \qquad \mathrm{E}[a_{\ell,m}(t) a_{\ell',m'}(s)] = \mathbb{1}_{\{\ell=\ell',\, m=m'\}} C_\ell(t-s), \tag{7.4}$$

with convergence in $L^2(\Omega \times \mathbb{S}^2)$ for every t. Each multipole $Z_\ell(\cdot, t) = \sum_m a_{\ell,m}(t) Y_{\ell,m}$ is a random eigenfunction of the spherical Laplacian with eigenvalue $-\ell(\ell+1)$, and we assume that $C_\ell(0) > 0$ and

$$\sum_{\ell \geq 0} \frac{2\ell+1}{4\pi} C_\ell(0) = 1, \tag{7.5}$$

so that $\mathrm{E}\, Z(x,t)^2 = 1$ by (7.4). Hence the sampled sequence $X_k \doteq Z(\cdot, k)$, $k \in \mathbb{Z}$, is centered, stationary and Gaussian with values in $\mathbb{H}_1$, and its covariance operator satisfies $QY_{\ell,m} = C_\ell(0) Y_{\ell,m}$ and $\mathrm{Tr}(Q) = 4\pi$.

Throughout this section, suppose that $\{X_k\}_{k \in \mathbb{Z}}$ satisfies Assumption 3.1 in the spherical harmonic basis. Index the input coordinates by $\mathcal{J} = \{(\ell, m) : \ell \geq 0,\ |m| \leq \ell\}$ through a fixed enumeration, and write $\ell(j) = \ell$, $Y_j = Y_{\ell,m}$ and $\lambda_j = C_\ell(0)$ for $j = (\ell, m)$. The standardized coordinate processes $\xi_j(k) = a_{\ell,m}(k)/\sqrt{C_\ell(0)}$ are independent. Since their laws depend only on ℓ, we write the parameters in Assumption 3.1 as $\alpha_j = 1 - \beta_{\ell(j)}$ and $\kappa_j = \kappa_{\ell(j)}$. Thus $\beta_\ell \in (0,1)$ is the memory parameter of the degree-ℓ coefficient processes, and $\kappa_\ell > 0$ is their standardized spectral scale, so that, for $j = (\ell, m)$,

$$\frac{f_j(\omega)}{C_\ell(0)} \sim \kappa_\ell |\omega|^{\beta_\ell - 1}, \qquad \omega \to 0.$$

In particular, $\beta_0 = 1 - \alpha_{(0,0)}$ is the memory parameter of $a_{0,0}$, the coefficient of the constant spherical harmonic, determining the monopole $Z_0(x,t) = a_{0,0}(t)/\sqrt{4\pi}$. Smaller values of β_ℓ correspond to stronger memory. Define

$$\beta_* \doteq \inf_{\ell \geq 1} \beta_\ell, \qquad \mathcal{I}^* \doteq \{\ell \geq 1 : \beta_\ell = \beta_*\}.$$

From Assumption 3.1, we have that $\inf_{\ell \geq 0} \beta_\ell = \min\{\beta_0, \beta_*\} > 0$. Denote by $\mathbf{e}_j$ the unit multiplicity index at $j \in \mathcal{J}$.

Let φ and Φ denote the standard Gaussian density and distribution function, respectively. For $u \in \mathbb{R}$, define

$$G_u[f] = \int_{\mathbb{S}^2} \mathbb{1}_{\{f(x) \geq u\}}\, dx, \qquad \overline{G}_u[f] = G_u[f] - 4\pi(1 - \Phi(u)), \qquad f \in \mathbb{H}_1.$$

For a Hermite multi-index $\boldsymbol{r} \in \mathcal{L}_p$, let $j_a = (\ell_a, m_a)$, $1 \leq a \leq p$, list its input coordinates, with each $j \in \mathcal{J}$ appearing exactly r_j times. Define the *generalized Gaunt integrals*

$$\mathcal{G}^{m_1 \cdots m_p}_{\ell_1 \cdots \ell_p} = \int_{\mathbb{S}^2} \prod_{a=1}^{p} Y_{\ell_a, m_a}(x)\, dx.$$

The following lemma identifies the Hermite expansion of $G = \overline{G}_u : \mathbb{H}_1 \to \mathbb{R}$.

Lemma 7.1. *For every $u \in \mathbb{R}$ and $k \in \mathbb{Z}$,*

$$\overline{G}_u[X_k] = \sum_{p=1}^{\infty} \sum_{\boldsymbol{r} \in \mathcal{L}_p} c_{u,\boldsymbol{r}} H_{\boldsymbol{r}}[X_k], \qquad c_{u,\boldsymbol{r}} = \frac{\varphi(u) H_{p-1}(u)}{\boldsymbol{r}!} \left(\prod_{a=1}^{p} C_{\ell_a}(0)^{1/2} \right) \mathcal{G}_{\ell_1 \cdots \ell_p}^{m_1 \cdots m_p}, \tag{7.6}$$

for $\boldsymbol{r} \in \mathcal{L}_p$, where the series converges in $L^2(\Omega)$ and $H_{\boldsymbol{r}}[X_k] = \prod_{j \in \mathcal{J}} H_{r_j}(\xi_j(k))$.

Proof. From Lemma 4.1 of [34], we can write

$$\begin{aligned} \overline{G}_u[Z(\cdot,k)] &= \sum_{p=1}^{\infty} \frac{\varphi(u) H_{p-1}(u)}{p!} \int_{\mathbb{S}^2} H_p(Z(x,k))\, dx \\ &= \sum_{p=1}^{\infty} \frac{\varphi(u) H_{p-1}(u)}{p!} \int_{\mathbb{S}^2} \sum_{\boldsymbol{r} \in \mathcal{L}_p} \frac{p!}{\boldsymbol{r}!} \left(\prod_{a=1}^{p} C_{\ell_a}(0)^{1/2} Y_{\ell_a, m_a}(x) \right) H_{\boldsymbol{r}}[X_k]\, dx, \end{aligned}$$

where the second line follows from decomposing $Z(x,k) = \sum_j C_{\ell(j)}(0)^{1/2} Y_j(x) \xi_j(k)$, the fact that $\sum_j C_{\ell(j)}(0) Y_j(x)^2 = 1$ by (7.5), and the multinomial Hermite formula. □

Recall from (3.3) that

$$\beta(\boldsymbol{r}) = \sum_{a=1}^{p} \beta_{\ell_a}, \qquad H(\boldsymbol{r}) = 1 - \tfrac{1}{2}\beta(\boldsymbol{r}).$$

Thus the active terms with the smallest $\beta(\boldsymbol{r})$ have the largest effective exponent $H(\boldsymbol{r})$ and determine the normalization. To describe the possible dominant cubic terms, define

$$\mathcal{C}_* = \left\{ \boldsymbol{r} \in \mathcal{L}_3 : \ell_1, \ell_2, \ell_3 \in \mathcal{I}^*, \quad \mathcal{G}_{\ell_1 \ell_2 \ell_3}^{m_1 m_2 m_3} \neq 0 \right\}.$$

We call u *admissible* if either (i) $u \neq 0$ and $\beta_0 \leq 2\beta_*$, (ii) $u \neq 0$ and $\mathcal{I}^* \neq \varnothing$, (iii) $u = 0$ and $\beta_0 \leq 3\beta_*$, or (iv) $u = 0$ and $\mathcal{I}^*$ contains an even integer. Under these four cases, the smallest effective memory parameter is attained by an active Hermite term. The following lemma identifies the corresponding exponents and dominant sets.

Lemma 7.2. *In the setting above, let u be admissible. Then $\overline{G}_u$ is centered, bounded and nonzero, and satisfies Assumption 3.2. For $u \neq 0$, its exponent and dominant set are*

$$d_u = 1 - \tfrac{1}{2}\min\{\beta_0, 2\beta_*\}, \quad \mathcal{D}_u = \begin{cases} \{\mathbf{e}_{(0,0)}\}, & \beta_0 < 2\beta_*, \\ \{2\mathbf{e}_{(\ell,m)} : \ell \in \mathcal{I}^*,\ |m| \leq \ell\}, & 2\beta_* < \beta_0, \\ \{\mathbf{e}_{(0,0)}\} \cup \{2\mathbf{e}_{(\ell,m)} : \ell \in \mathcal{I}^*,\ |m| \leq \ell\}, & \beta_0 = 2\beta_*. \end{cases} \tag{7.7}$$

For $u = 0$, they are

$$d_0 = 1 - \tfrac{1}{2}\min\{\beta_0, 3\beta_*\}, \quad \mathcal{D}_0 = \begin{cases} \{\mathbf{e}_{(0,0)}\}, & \beta_0 < 3\beta_*, \\ \mathcal{C}_*, & 3\beta_* < \beta_0, \\ \{\mathbf{e}_{(0,0)}\} \cup \mathcal{C}_*, & \beta_0 = 3\beta_*. \end{cases} \tag{7.8}$$

Moreover, $\mathcal{C}_* \neq \varnothing$ if and only if $\mathcal{I}^*$ contains an even integer.

Proof. First, $|\overline{G}_u| \leq 4\pi$ and $\mathrm{E}\,\overline{G}_u[X_0] = 0$. We identify the dominant terms by comparing their effective exponents $H(\boldsymbol{r}) = 1 - \beta(\boldsymbol{r})/2$ in the different parameter regimes. We then verify Assumption 3.2 and explain why the remaining chaoses are negligible under the resulting normalization. By orthonormality,

$$\mathcal{G}_{\ell}^{m} = \sqrt{4\pi}\, \mathbb{1}_{\{(\ell,m)=(0,0)\}}, \qquad \mathcal{G}_{\ell_1 \ell_2}^{m_1 m_2} = \mathbb{1}_{\{\ell_1 = \ell_2,\ m_1 = m_2\}},$$

implying that the only active first-order index is $\mathbf{e}_{(0,0)}$, and the only possible second-order indices are $2\mathbf{e}_{(\ell,m)}$, with coefficients

$$c_{u,\mathbf{e}_{(0,0)}} = \varphi(u)\sqrt{4\pi C_0(0)}, \qquad c_{u,2\mathbf{e}_{(\ell,m)}} = \frac{u\varphi(u)C_\ell(0)}{2}. \tag{7.9}$$

The first coefficient is nonzero for every u, so $\overline{G}_u$ is nonzero and has an active term with effective exponent $1-\beta_0/2$. Moreover, any index $\boldsymbol{r}$ of order $p \geq 2$ involving the monopole satisfies $\beta(\boldsymbol{r}) \geq \beta_0 + (p-1)\min\{\beta_0, \beta_*\} > \beta_0$. Its effective exponent is therefore strictly smaller than that of the linear monopole term, in every parameter regime.

We first consider $u \neq 0$. All the quadratic coefficients in (7.9) are nonzero. For $\ell \geq 1$, we have that $H(2\mathbf{e}_{(\ell,m)}) = 1-\beta_\ell \leq 1-\beta_*$, and equality holds when $\ell \in \mathcal{I}^*$. Thus the competition is between the monopole exponent $1-\beta_0/2$ and the supremum $1-\beta_*$ of the quadratic exponents over positive degrees. If $\beta_0 < 2\beta_*$, the monopole exponent is larger. If $2\beta_* < \beta_0$, the quadratic exponent is larger and attained on $\mathcal{I}^* \neq \varnothing$. If $\beta_0 = 2\beta_*$, the monopole and all quadratic terms with $\ell \in \mathcal{I}^*$ have the same effective exponent and must be retained together; if $\mathcal{I}^* = \varnothing$, only the monopole attains it.

To exclude higher chaoses, it remains to consider indices that do not involve the monopole. For every such index of order $p \geq 3$, $\beta(\boldsymbol{r}) \geq p\beta_* \geq 3\beta_* > 2\beta_*$, meaning $H(\boldsymbol{r}) < 1-\beta_* \leq d_u$. Together with the preceding bound for indices involving the monopole, this shows that no higher-order term can the dominant exponent. The three comparisons above therefore give (7.7).

We now consider $u = 0$. Every even chaos vanishes because $H_{p-1}(0) = 0$ for even p, whereas the first-order coefficient remains nonzero. Since $H_2(0) = -1$, a cubic coefficient is nonzero exactly when $G^{m_1m_2m_3}_{\ell_1\ell_2\ell_3} \neq 0$. For cubic indices not involving the monopole, $H(\boldsymbol{r}) = 1-\frac{1}{2}(\beta_{\ell_1}+\beta_{\ell_2}+\beta_{\ell_3}) \leq 1-\frac{3}{2}\beta_*$, and equality holds precisely for the active indices in $\mathcal{C}_*$. To determine when this set is nonempty, recall that $Y_{\ell,m}(-x) = (-1)^\ell Y_{\ell,m}(x)$. Hence a Gaunt integral vanishes whenever the sum of its degrees is odd. In particular, $\mathcal{C}_* = \varnothing$ if every degree in $\mathcal{I}^*$ is odd. The converse is also true, since for every even degree ℓ, the standard zonal harmonic satisfies $\mathcal{G}^{000}_{\ell\ell\ell} > 0$, so $\mathcal{C}_* \neq \varnothing$ whenever $\mathcal{I}^*$ contains an even integer.

We therefore compare the monopole exponent $1-\beta_0/2$ with the cubic bound $1-3\beta_*/2$. If $\beta_0 < 3\beta_*$, the monopole dominates every cubic term. If $3\beta_* < \beta_0$, admissibility ensures that $\mathcal{C}_* \neq \varnothing$, and precisely these cubic terms attain the larger exponent. If $\beta_0 = 3\beta_*$, the monopole and the terms in $\mathcal{C}_*$ have equal effective exponents, with only the monopole retained when $\mathcal{C}_* = \varnothing$. Every remaining active cubic index has a smaller effective exponent than d_0. Finally, for indices of odd order $p \geq 5$ not involving the monopole, $\beta(\boldsymbol{r}) \geq p\beta_* \geq 5\beta_* > 3\beta_*$, so that $H(\boldsymbol{r}) < 1-\frac{3}{2}\beta_* \leq d_0$ proving (7.8).

In both cases, the smallest effective memory parameter is positive and at most $\beta_0 < 1$, so $1/2 < d_u < 1$. We can, then conclude that $\mathcal{D}_u \neq \varnothing$, and, moreover,

$$\frac{\mathrm{E}\,|\overline{G}_u[X_0]|^2}{2d_u-1} \leq \frac{16\pi^2}{2d_u-1} < \infty,$$

so Assumption 3.2 is satisfied. □

7.2.2 The noncentral limit theorem

Before stating the result, we describe the limiting processes, whose values at $s=1$ give the limiting random variables for the normalized excursion sums. On the common spectral probability space of Section 2.2, for $\boldsymbol{r} \neq 0$ with $\beta(\boldsymbol{r}) < 1$ and $s \in [0,1]$, set

$$\mathcal{Z}_{\boldsymbol{r}}(s) = \kappa^{\boldsymbol{r}/2} I_{|\boldsymbol{r}|}(J_{\boldsymbol{r}} K_{s,\boldsymbol{r}}), \qquad \kappa^{\boldsymbol{r}/2} = \prod_j \kappa_{\ell(j)}^{r_j/2},$$

where $J_{\boldsymbol{r}}$ and $K_{s,\boldsymbol{r}}$ are defined in (2.7) and (4.2), respectively. Define

$$\mathcal{B}_u(s) = \varphi(u)\sqrt{4\pi C_0(0)}\,\mathcal{Z}_{\mathbf{e}_{(0,0)}}(s),$$
$$\mathcal{R}_u(s) = \frac{u\varphi(u)}{2}\sum_{\ell\in\mathcal{I}^*} C_\ell(0)\sum_{m=-\ell}^{\ell}\mathcal{Z}_{2\mathbf{e}_{(\ell,m)}}(s), \qquad 2\beta_* < 1,$$
$$\mathcal{T}(s) = -\varphi(0)\sum_{\boldsymbol{r}\in\mathcal{C}_*}\frac{\mathcal{G}^{m_1m_2m_3}_{\ell_1\ell_2\ell_3}}{\boldsymbol{r}!}\left(\prod_{a=1}^{3} C_{\ell_a}(0)^{1/2}\right)\mathcal{Z}_{\boldsymbol{r}}(s), \qquad 3\beta_* < 1,$$

with the convention that, the sum over an empty set vanishes. These series converge in $L^2(\Omega)$ for every s, by the labelled isometry (2.8), the uniform kernel bound (5.1), and $\sum_{\boldsymbol{r}}\boldsymbol{r}!|c_{u,\boldsymbol{r}}|^2 = \mathrm{E}\,|\overline{G}_u[X_0]|^2 < \infty$.

The process $\mathcal{B}_u$ is a multiple of a fractional Brownian motion with Hurst index $1-\beta_0/2$; the process $\mathcal{R}_u$ is a weighted sum of independent Rosenblatt processes with Hurst index $1-\beta_*$, which follows from their first- and second-order spectral kernels, respectively; If $\mathcal{C}_* \neq \varnothing$, then $\mathcal{T}$ is a nonzero process in the third Wiener chaos Moreover, $\mathcal{B}_u$ depends only on $\widehat{W}_{(0,0)}$, whereas $\mathcal{R}_u$ and $\mathcal{T}$ depend only on $\{\widehat{W}_{(\ell,m)} : \ell \in \mathcal{I}^*,\ |m| \le \ell\}$, so that $\mathcal{B}_u$ is independent of the latter processes.

For an admissible threshold, define

$$\mathcal{E}_u = \begin{cases} \mathcal{B}_u, & \beta_0 < 2\beta_*, \\ \mathcal{R}_u, & 2\beta_* < \beta_0,\ u \neq 0, \\ \mathcal{B}_u + \mathcal{R}_u, & \beta_0 = 2\beta_*, \end{cases} \qquad \mathcal{E}_0 = \begin{cases} \mathcal{B}_0, & \beta_0 < 3\beta_*, \\ \mathcal{T}, & 3\beta_* < \beta_0, \\ \mathcal{B}_0 + \mathcal{T}, & \beta_0 = 3\beta_*. \end{cases} \tag{7.10}$$

The conditions for defining $\mathcal{R}_u$ and $\mathcal{T}$ hold whenever these processes occur in (7.10), since $\beta_0 < 1$. In the equality cases, the two components are independent; the limit is non-Gaussian whenever the higher-order component is nonzero.

Theorem 7.1. *Let $X_k = Z(\cdot,k)$ be the spherical Gaussian sequence above, satisfying Assumption 3.1 in the spherical harmonic basis. Let $u_1,\dots,u_r$ be admissible thresholds, and set $G = (\overline{G}_{u_1},\dots,\overline{G}_{u_r})$ and $D = \mathrm{diag}(d_{u_1},\dots,d_{u_r})$, with the exponents of Lemma 7.2. Then, as $n\to\infty$,*

$$n^{-D}\sum_{k=1}^{\lfloor ns\rfloor} G[X_k] \xrightarrow{f.d.d.} (\mathcal{E}_{u_1}(s),\dots,\mathcal{E}_{u_r}(s)).$$

Proof. By Lemma 7.2, each coordinate satisfies item (i) of Assumption 3.2, and

$$\sum_{i=1}^{r}\frac{\mathrm{E}\,|\overline{G}_{u_i}[X_0]|^2}{2d_{u_i}-1} < \infty, \qquad \min_{1\le i\le r} d_{u_i} > \tfrac{1}{2}.$$

Theorem 3.1 therefore yields both conclusions. Its ith limit coordinate is $\sum_{\boldsymbol{r}\in\mathcal{D}_{u_i}} c_{u_i,\boldsymbol{r}}\mathcal{Z}_{\boldsymbol{r}}$, which equals $\mathcal{E}_{u_i}$ by (7.6) and the dominant sets in Lemma 7.2. □

We now present a couple of remarks regarding our theorem

Remark 7.1. If $u \neq 0$, $2\beta_* < \beta_0$ and $\mathcal{I}^* = \varnothing$, the supremum defining d_u is not attained, so our main theorem does not apply. For $u = 0$ outside the admissible regimes, the active indices must be determined from (7.6). Whenever $\min_{c_{0,\boldsymbol{r}}\neq 0}\beta(\boldsymbol{r})$ is attained, the main theorem applies with d_0 equal to one minus half this minimum.

The next remark compares our result with that of [34].

Remark 7.2. We now compare our result with those of [34], first at the level of conclusions and then in terms of the assumptions on the process $\{X_k\}$.

In the notation of [34], $\mathcal{A}_u(t) = G_u[Z(\cdot,t)]$, and their asymptotic results concern the continuous-time integral

$$\mathcal{M}_T(u) = \int_0^T \big(\mathcal{A}_u(t) - \mathrm{E}\,\mathcal{A}_u(t)\big)\,dt = \int_0^T \overline{G}_u[Z(\cdot,t)]\,dt.$$

Their limits are stated after variance normalization, whereas our normalization uses powers n^{-d_u}. Their Theorems 2.2 and 2.7 establish convergence to Gaussian and composite Rosenblatt random variables, respectively, with the distributional conclusion of Theorem 2.7 assuming finitely many dominant multipoles. Their Theorem 2.9 establishes a reduction to the third-chaos component for $u = 0$, but does not establish its limiting distribution; see Remark 2.10 therein.

Our Theorem 7.1 concerns sampled partial-sum processes and provides discrete-time counterparts of these long-memory results. Passing from sampled sums to continuous-time integrals would require a separate approximation argument, which we do not provide here. Within the corresponding long-memory regimes, our theorem provides the following additional conclusions: (i) we identify the third-chaos process $\mathcal{T}$ explicitly and prove convergence to it; (ii) we do not require finiteness of $\mathcal{I}^*$ or $\mathcal{C}_*$; (iii) we obtain joint finite-dimensional convergence at finitely many thresholds $u_1, \dots, u_r$, allowing different normalization exponents; and (iv) we treat the equality cases $\beta_0 = 2\beta_*$ and $\beta_0 = 3\beta_*$, discussed in Remark 2.11 of [34].

The assumptions in [34] are formulated in the time domain and are therefore not immediately comparable to our spectral Assumption 3.1. One difference is that our assumption requires every retained multipole to have long memory, so the short-memory and critical logarithmic regimes considered in [34] are not covered by our theorem.

Nevertheless, Remark 3.3 allows us to verify Assumption 3.1 for a subclass of their covariance family. Within the setting presented in Condition 1.3 [34], suppose that

$$C_\ell(\tau) = G_\ell(\tau)(1+|\tau|)^{-\beta_\ell}, \qquad 0 < \inf_{\ell\ge 0}\beta_\ell \le \sup_{\ell\ge 0}\beta_\ell < 1,$$

where $C_\ell(0) > 0$, and additionally assume that $\sup_{\ell\ge 0}\sum_{h=1}^{\infty}\left|\frac{G_\ell(h)}{C_\ell(0)} - 1\right| < \infty$. These conditions are stronger than Condition 1.3 [34] since $\sup_\ell \beta_\ell < 1$ and (ii) they require the absolute summability of $\left|\frac{G_\ell(h)}{C_\ell(0)} - 1\right|$. Under these conditions, the conditions of Remark 3.3 hold, and so Assumption 3.1 follows. We stress that these additional conditions are sufficient, rather than necessary, for our spectral assumption.

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